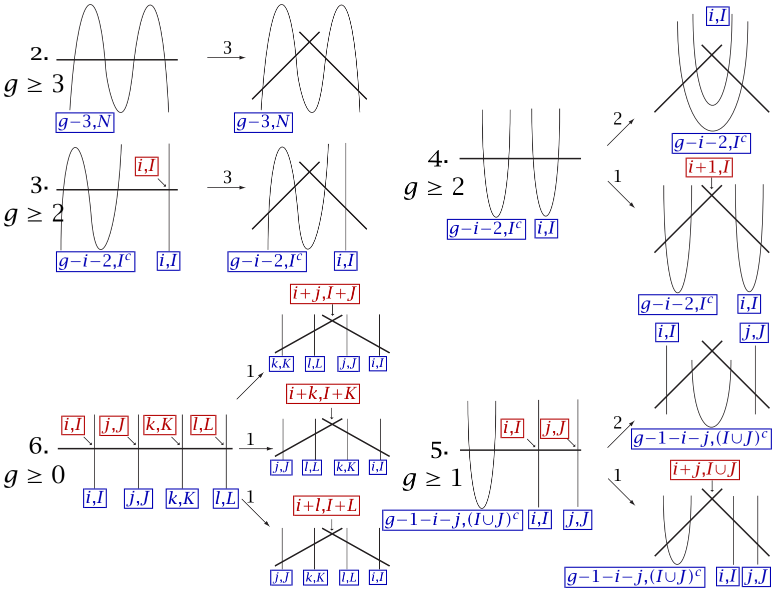

Mori Theory of Moduli Spaces of Stable Curves

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Preface

This is an working draft of a set of lecture notes on the birational geometry of moduli spaces of stable curves.

This is a working draft of February 10, 2010. The most recent draft of these notes may be downloaded at http://projectivepress.com/moduli/moristablecurves.pdf.

Comments, corrections and other suggestions for improving these notes are welcome. Please email them to me at morrison@fordham.edu.

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Introduction

These notes will try to introduce the reader to a range of questions dealing with the birational geometry of moduli spaces of pointed stable curves (and, although only in passing, maps) particularly the Mori theory of their standard models. The guiding philosophy, popularized by Mumford, is that, not only it is fair game to ask about moduli spaces any question that is of interest for a general variety, but that the modular property—by letting apply results about families—will often allow us to say more about the answers for moduli spaces than we can hope to in general. In recent years, a number of results have also demonstrated that the web of relationships that links different moduli spaces, exemplified by the inductive structure of the stratifications of the spaces $\overline{M}_{g,n}$ by topological type, can make it easier to proving results for the right collections of moduli spaces easier than proving them for the individual spaces.

I have tried to make the notes as user-friendly as possible, to assume only a basic familiarity with the language and actors of rmoduli spaces of stable curves and of Mori theory, and to recall basic definitions and results as they are needed. The previous sentence is, of course, a bald-faced lie. The reader for whom all this pre-requisite material is genuinely new will almost certainly be at sea fairly quickly without some further background. For basic material about moduli spaces of curves, *Moduli of Curves* [32] will be an indispensable reference. Indeed, I have borrowed a number of figures and many notations from that book and I have provided, as an appendix, a list of the typos in *Moduli of Curves* [32] of which I am aware—surely not all. For the facts about Mori theory that I need, the basic reference is [47].

I have also tried to provide a look at both classic (defined as, say, from the last millenium) and recent work and to give some sense of how our picture of these questions has evolved over the last roughly 30 years. As a result, each lecture reviews many results and it has not been possible in four talks to prove everything even in outline. However, I also tried to provide more than a roster of definitions and statements and to explain one or more key steps in the proof of each main result. Even so, the reader who wants to understand the area is perhaps best advised to treat these notes as a set of annotated suggestions for further reading.

Here is a brief summary of what *is* discussed in each of the lectures. The first lecture reviews background about the spaces themselves, the divisor classes they carry and basic techniques for working with them. The second deals with bounds for the cone of effective divisors from inside (the Eisenbud-Harris-Mumford computation of the classes of Brill-Noether divisors and its generalization to Koszul divisors by Farkas), outside (my work with Harris on slope inequalities for effective divisors) and both (Keel and McKernan's description of the effective cone of $\overline{M}_{0,\tilde{n}}$ leading to the the description for $\overline{M}_{0,0}(\mathbb{P}^d, d)$ recently given by Coskun, Harris and Starr and, independently, by Keel). Lecture three treats analogous questions for ample and nef divisors reviewing constructions of such divisors (due to Parshin, Knudsen and Mumford, Cornalba and Harris, and Farkas and Gibney), inequalities bounding the ample cone (due to Keel and McKernan, to Faber, and to Gibney, Keel and myself) and results comparing these cones for different moduli spaces that arise in producing inequalities and in work of Coskun, Harris and Starr on the ample cone of $\overline{M}_{0,0}(\mathbb{P}^d, d)$. A final lecture—not included in this version—will study log canonical models of the pair $(\overline{\mathcal{M}}_g, \Delta)$, reviewing both initial results of Hassett and Hyeon [36] for general g based on an old construction of Schubert [58], more complete results in genera 2 (Hassett [35]) and 3 (Hyeon and Lee [38]) and ongoing work of Smyth [59] on $\overline{M}_{1,n}$.

Here also are a few disclaimers about what the reader will *not* find in these notes. First, the talks were designed to introduce non-specialists to the subject and much of their audience came from the Teichmüller side of the subject I have adopted the "fat chance" approach to stack-theoretic issues. This means minimizing (if not *quite* eliminating) references to moduli stacks and stacky variants of spacey concepts and has the advantage of making *Moduli of Curves* [32] a more appropriate reference. The price paid is the need for minor but inelegant book-keeping in many calculations and for some major arm-waving at a few points. The second major lacuna, dictated partly by time (these notes grew out of two short lecture series) but also in some ways a corollary of the first, is that there is almost no discussion of moduli spaces of stable maps. I made an modest exception to mention the work of Coskun, Harris and Starr cited above both because of its elegance and because of their novel use of test families. Finally, I make no pretense to the completeness of this survey. Many other results are missing not because I did not want to include them but because I could not squeeze them, even allusively, onto my critical path.

One more remark I'd like to make here, since it remains in the background in above and, to an extent, in the notes, are the signs of life from the dead parrot[1] of geometric invariant theory. All the constructions of ample classes in lecture three and of log canonical models in lecture four depend on stability and instability results for Hilbert schemes of n-canonical curves for values of n less than the minimum

[1]The Monty Python reference is to a survey talk on applications of GIT to constructions of moduli that I gave at an American Institute of Mathematics Workshop on *Compact moduli* in which I tried to uphold the Cleesian thesis that the parrot was only sleeping against János Kollár's Palinian view that it had been done in by the modern minimal model program

5 that can be used in Gieseker's GIT construction of $\overline{M}_g$. Ironically, the most subtle GIT questions arise in carrying out log minimal model program for $(\overline{\mathcal{M}}_g, \delta)$.

An early version of these lectures was given at the *EAGER Summer School on Moduli Spaces of Curves* held in Trento, Italy in September 2001. Since then, there has been much activity in this area. These notes are a revision and extension presented at the Centre de Récherches Mathématiques in Montréal in June, 2007 as part of its *Workshop on Moduli Spaces and Related Topics*. I'd like to thank the organizers of both meetings—Giorgio Bolondi and Ciro Ciliberto for the EAGER group and Marco Bertola and Dmitri Korotkin for the CRM—for providing the impetus to develop these notes. A Fordham University Faculty Fellowship enabled me to visit the Tata Institute of Fundamental Research and the University of Sydney in 2007-2008. Both institutions provided excellent environments in which to revise my notes and I'd like to thank Vasudevan Srinivas and Gus Lehrer, respectively, for their help in arranging these visits. Finally, many mathematicians have helped me by explaining their ideas, supplying early drafts of work-in-progress or in other ways and I wish to take this opportunity to thank Enrico Arbarello, Dave Bayer, Carel Faber, Gavril Farkas, Angela Gibney, Brendan Hassett, Joe Harris, Donghoon Hyeon, Seán Keel, Jason Starr, Michael Thaddeus, Ravi Vakil and Angelo Vistoli.

Chapter 1

Preliminaries concerning divisor classes

The aim of this lecture is to take a first look at the most natural divisor classes on moduli spaces of n-pointed stable curves, or, *curves* as we'll usually refer to them for convenience. We also want to review the basic methods for obtaining relations amongst divisor classes and illustrate their use by deriving some basic relations which we'll need in the later lectures. Since there's a lot of material to cover and most of it is explained in detail in *Moduli of Curves* [32] , I will often simply quote prerequisites.

1.1 Basic divisor classes

I'll start by defining the most important divisor classes on moduli spaces of stable curves—we'll find generators and relations for the Picard groups of all the spaces $\overline{M}_{g,n}$—and reviewing how they behave under the natural maps that relate these spaces. In the introductory spirit of these lectures, I'll review some very basic definitions but with the hope that I am recalling well-known facts for most of you.

The Hodge class

Let's begin with the moduli space M_g of ordinary smooth curves which comes equipped with one obvious line bundle. The the universal curve $C_g \longrightarrow M_g$ comes equipped with a relative dualizing sheaf ω_{C_g/M_g} which we can think of more naively as the bundle whose restriction to each fiber C is the canonical bundle. Taking the direct image of this bundle gives a bundle of rank g on M_g whose fiber over $[C]$ is just $H^0(C, K_C)$.

Definition 1.1: We call this bundle the Hodge bundle and denote it by Λ. We set

$$\lambda_i = c_i(\Lambda)$$

and call the divisor class $\lambda = \lambda_1$ the Hodge class.

There is one small problem with this definition: a universal curve C_g exists only over the locus M_g^0 of curves without automorphisms where we have fine moduli. Since it turns out that for $g \geq 2$ curves with automorphisms have codimension at least $g - 2$ in M_g this allows us to make sense of the class λ for $g \geq 3$.

We'll deal with this issue later but for now I'd like focus on introducing the cast so I'm going to pretend that all moduli spaces are fine and will continue to imagine that there is a universal curve $\pi : C_g \longrightarrow M_g$ with relative dualizing sheaf $\omega = \omega_{C_g/M_g}$.

Note that there is a second way to use ω to produce classes on M_g. Instead of first pushing down to M_g and then taking a Chern class, we can reverse the order of these operations. Define

$$\gamma = c_1(\omega)$$

which is a divisor class on C_g and then set

$$\kappa_1 := \pi_*(\gamma^2)$$

(the squaring produces a class in codimension 2 on C_g which then pushes down to one of codimension 1 in M_g). Likewise, we can set

$$\kappa_i := \pi_*(\gamma^{i+1}) .$$

We'll see how these are related shortly.

It's not obvious that the class λ is even non-trivial but by topological methods coming from Teichmüller theory which we won't enter into at all, Harer has shown (among many other striking results),

HARER'S THEOREM 1.2 *For $g \geq 3$,*

$$H^1(M_g, \mathbb{Q}) = H^3(M_g, \mathbb{Q}) = 0$$

and

$$H^2(M_g, \mathbb{Q}) = \mathbb{Q} \cdot \lambda\,.$$

COROLLARY 1.3 *For $g \geq 3$, $\mathrm{Pic}(M_g) \bigotimes \mathbb{Q} = \mathbb{Q} \cdot \lambda$.*

For $g = 2$, things are rather different. We can define the class λ but only by the methods to be discussed later since every curve has an automorphism. Moreover, a theorem of Igusa says that M_2 is affine, hence, in this case, λ *is* trivial (although, as we'll see, this itself has some interesting consequences).

Since, we'll always be working modulo torsion, we'll drop the explicit references to tensoring Picard groups with $\mathbb{Q}$ in the sequel and write $\mathrm{Pic}(M)$ instead of $\mathrm{Pic}(M) \bigotimes \mathbb{Q}$ for all spaces M that arise.

Stable curves

Of course, to have an interesting divisor theory we need to work on a complete model and pass from M_g to $\overline{M}_g$, the moduli space of stable curves.

DEFINITION 1.4: A curve is called *stable* if it is complete, connected, has only ordinary nodes as singularities, and has finitely automorphism group. In view of the connectedness, a nodal curve can have infinite automorphism group only if it has rational components. So we can give alternate formulations as:

- every smooth rational component of C meets the other components in at least 3 points.
- every smooth rational component of the normalization of C has at least 3 points lying over singular points of C.

If we replace the number 3 by 2 in either of these definitions, the resulting curves are called semi-stable. By contracting rational curves meeting the other components in two points, every semi-stable curve determines a unique stable curve called its stable model.

For reference, we note a few basic facts about stable curves. For proofs, see Chapter 3 of *Moduli of Curves* [32].

Genus Formula 1.5 *Fix a connected nodal curve C of genus g with γ components C_j of genera g_j and δ nodes p_i and let $\nu : \tilde{C} \to C$ be its normalization and q_i and r_i be the points of $\tilde{C}$ lying over the node p_i. Then we have a long exact sequence*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{\tilde{C}} \longrightarrow \sum_{i=1}^{\delta} \mathbb{C}_{p_i} \longrightarrow 0$$

and the associated long exact sequence yields the genus formula

$$(g-1) = \sum_{j=1}^{\gamma} (g_j - 1) + \delta \quad \text{or} \quad g = \left(\sum_{j=1}^{\gamma} g_j \right) + \delta - \nu + 1 .$$

If $\varphi : C \to B$ is a flat family of connected nodal curves with smooth generic fiber, the relative dualizing sheaf $\omega_{C/B}$ is the unique line bundle on C which agrees with the relative cotangent bundle

$$\text{Coker}\big(d\varphi : \varphi^*(\Omega_B) \longrightarrow \Omega_C\big)$$

away from the singular locus of φ.

A local calculation shows that such an extension exists (and, since any two agree up to codimension 2, is unique) and that:

Ω **FORMULA 1.6** *If $\varphi : C \longrightarrow B$ is a one-parameter family of connected nodal curves over with smooth base B, smooth total space C, and smooth general fiber and $\mathcal{I}_Z$ is the ideal sheaf of the locus Z of nodes of fibers, then*

$$\Omega_{C/B} = \omega_{C/B} \bigotimes \mathcal{I}_Z .$$

We may also informally characterize $\omega_{C/B}$ as *the bundle whose sections restrict to rational differentials with "cancelling residues" on every fiber*. If the total space C is smooth, this simplifies to give $\omega_{C/B} = K_C \bigotimes (\varphi^*(K_B)^\vee)$. From this second characterization it follows that

$$R^1\varphi_*(\omega_{C/B}) = \mathcal{O}_B$$

and hence that the relative or Grothendieck duality formula

$$R^1\varphi_*(\omega_{C/B} \bigotimes L^\vee) = (\varphi_* L)^\vee$$

holds for bundles L on C for which $h^0(C_b, L|_{C_b})$ is constant. Finally, the relative dualizing sheaf is functorial in the sense that the relative dualizing sheaf of a pullback is the pullback of the relative dualizing sheaf. In particular, this means that we can use the relative dualizing sheaf $\omega_{\overline{C}_g/\overline{M}_g}$ of the universal curve $\overline{C}_g \longrightarrow \overline{M}_g$ to define a bundle Λ which extends the corresponding bundle on C_g (modulo, as usual, the fact that a universal curve exists only over the locus of stable curves without automorphisms). By pushing down and taking Chern classes (or vice-versa), we can then define classes λ_i and κ_i extending those defined on M_g above.

Finally, stable curves have moduli.

THEOREM 1.7 *A coarse moduli space $\overline{M}_g$ exists for stable curves of genus g: it is projective and irreducible.*

How does the Picard group of $\overline{M}_g$ differ from that of M_g? Any extra classes must live on the boundary

$$\Delta := \overline{M}_g \setminus M_g$$

—the locus of curves which are singular—but how many are there. The answer comes from the deformation theory of nodal curves. The key fact is:

Lemma 1.8 *A neighborhood of a nodal curve* $[C]$ *in its space of first order deformations may be identified with with a neighborhood of the origin in* $\mathbb{C}^{3g-3}$ *in such a way that the deformations of* C *preserving any node are identified with a smooth divisor, any collection of these divisors meet transversely at the origin and the normal space to any of the divisors at the origin is identified with the tensor product of the tangent spaces to the branches of* C *at the corresponding node.*

By the universal property of $\overline{M}_g$ this translates directly to:

Proposition 1.9 [Normal bundle to Δ] *In a neighborhood of the moduli point* $[C]$ *of a stable curve without automorphisms, the boundary* Δ *is a normal-crossings divisor, with branches corresponding one-to-one to the nodes of* C *and with the normal space to each branch isomorphic to the tensor product of the tangent spaces to the branches of* C *at the corresponding node.*

This fact will be essential to making enumerative calculations later on. We can draw one conclusion immediately: *All singular stable curves lie in the closure of the locus of stable curves with a single node.*

What are the possibilities for a curve of genus g with a single node? By the genus formula, such a curve is either irreducible or the union of irreducible curves of genera i and $g - i$ meeting at a single point. We denote the closures of these loci by Δ_{irr} and $\Delta_i, i = 1, \ldots, \lfloor g/2 \rfloor$. We can represent generic curves in the two components of Δ in $\overline{M}_3$ schematically with either of the sets of pictures
or
The second set is actually less misleading since the branches at the nodes are shown as transverse.

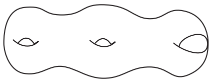


FIGURE 1.10: Δ_{irr} and Δ_1 in $\overline{M}_3$—surface sketches

FIGURE 1.11: Δ_{irr} and Δ_1 in $\overline{M}_3$—schematic sketches

PROPOSITION 1.12 *The closures of the loci Δ_{irr} and Δ_i are the irreducible components of Δ.*

One the one hand, the intersection of any two of these loci consists of curves with at least two nodes so has codimension 2 in $\overline{M}_g$ so each is a union of components. To see that each is actually irreducible, we need only exhibit dominating maps to each from irreducible varieties which we'll do in the next section using moduli spaces of pointed curves. Adopting the convention that the divisor class defined by a boundary class Δ (possibly with decoration) is denoted by δ (similarly decorated), we can immediately conclude that

COROLLARY 1.13 *For $g \geq 3$,*

$$\mathrm{Pic}(\overline{M}_g) = \mathbb{Q} \cdot \lambda \oplus \mathbb{Q} \cdot \Delta_{\mathrm{irr}} \oplus \Big(\bigoplus_{i=1}^{\lfloor g/2 \rfloor} \mathbb{Q} \cdot \Delta_i \Big).$$

For $g = 2$, the boundary classes alone generate since $\mathrm{Pic}(M_2)$ *is trivial.*

One further remark before we close this section. A component of a stable curve which has genus 1 and meets the rest of the curve in a single point is called an elliptic tail. Such a curve always has a nontrivial automorphism—the involution on the elliptic tail fixing the join point. Thus Δ_1 is a divisor in $\overline{M}_g$ consisting entirely of curves with automorphisms. This means it's always an exception in enumerative calculations. But, it is exceptional in another way: at generic points of Δ_1 the automorphism group is exactly $\mathbb{Z}/2\mathbb{Z}$ so $\overline{M}_g$ is smooth at such points. The quotient by an involution which extends to a divisor is the exception to the principle that $\overline{M}_g$ is singular at curves with automorphisms: the only other example is the hyperelliptic locus in genus 3.

Pointed stable curves

Definition 1.14: A stable *n*-pointed curve $(C, p_1, \dots, p_n)$ is a complete connected nodal curve together with an *ordered* choice of n *distinct smooth* points (called the marked points of C) such that the group of automorphisms of C fixing the marked points is finite (or equivalently, such that every smooth rational component contains at least three nodes or marked points). It's convenient to allow the marked points to be indexed by *any* finite ordered set N of cardinality n, not just the standard one $\mathbf{n} = \{1, 2, \dots, n\}$ (in practice, such an N is usually a subset of a standard one). We then speak of N-pointed stable curves.

Once again, the basic fact about moduli of such curves is

Theorem 1.15 *A coarse moduli space $\overline{M}_{g,N}$ exists for N-pointed stable curves of genus g: it is projective and irreducible.*

For unpointed curves, Gieseker's GIT construction [26, 27] is discussed in Chapter 4 of *Moduli of Curves* [32]. Only recently has the extension—surprisingly non-trivial—of this construction to pointed curves (see [3]) been undertaken. An alternate approach in the spirit of the log-minimal model program can be found in [34]. One point to emphasise is that these spaces are also defined for $g < 2$: for $g = 1$, just a single marked point is needed to kill the one-parameter group of automorphisms given by translation, while for $g = 0$, we need at least 3 marked points.

As an example, let's look at the $M_{0,n}$ for small n. By the sharp triple transitivity of PGL(2), $\overline{M}_{0,3}$ is a point and $\overline{M}_{0,4}$—which is also the universal curve *over* $\overline{M}_{0,3}$—is $\mathbb{P}^1$ with the points 0, 1 and ∞ marked. (Of course, we could equally well mark any other set of three points but this is the natural and standard choice). What's the universal curve over $\overline{M}_{0,4}$? An obvious candidate is the surface $\mathbb{P}^1 \times \mathbb{P}^1$ marked by the constant sections $\{0\} \times \mathbb{P}^1$, $\{1\} \times \mathbb{P}^1$, and $\{\infty\} \times \mathbb{P}^1$ and the diagonal. There's a small problem: over the points 0 , 1 and ∞ in $\overline{M}_{0,4}$, the diagonal meets the corresponding constant section and we don't have 4 distinct marked points. This is easily remedied by blowing up these three points of intersection. So we see that there are three singular curves in $\overline{M}_{0,4}$, each of which consists of a pair of $\mathbb{P}^1$'s with two marked points meeting in a single node with each division of the three divisons of the 4 marked points into two pairs occuring once. The picture near such a fiber before and after the blowup is as shown on the left and right respectively in Figure 1.16.

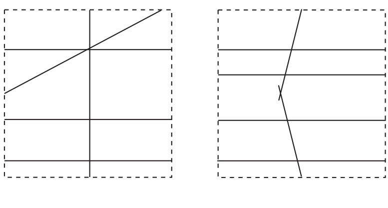


FIGURE 1.16: Singular fibers of the universal curve over $\overline{M}_{0,4}$

EXERCISE 1.17: Find the next stage in this tower. That is, show that $\overline{M}_{0,j}5$ is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in the three points $(0,0)$, $(1,1)$, and (∞,∞) and express the universal curve over it as a blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The full story in is [40].

Let's do one more simple example. The space $M_{1,1}$ is the moduli space of elliptic curves (that is, curves of genus 1 *plus* a marked origin) which is just the j-line, a copy of $\mathbb{A}^1$ parameterized by the j-invariant of the curve so $\overline{M}_{1,1} \cong \mathbb{P}^1$. The universal curve here is then some rational elliptic surface with having singular fibers over the pole of the j-function at ∞. In this case, there can be no reducible singular fiber: the only possibility permitted by the genus formula is a curve of genus 1 meeting a rational curve in a single node but since there is only one marked point such a rational curve can be at best semi-stable.

Definition 1.18: In various situations, it is necessary to "forget" the all or part of the ordering of the set of marked points, or equivalently to take a quotient $\overline{M}_{g,n}/\mathfrak{G}$ of $\overline{M}_{g,n}$ by a subgroup $\mathfrak{G}$ of the symmetric group $\mathfrak{S}_n$. If, as will usually be the case, $\mathfrak{G}$ fixes the last m of $n = l + m$ points, we will write $n = l + \widetilde{m}$. Thus $\overline{M}_{g,\widetilde{n}}$ denotes the quotient by the full symmetric group and $\overline{M}_{g,l+\widetilde{m}}$ the quotient by the group permuting the last m points.

There are three natural collections of maps which connect the family of *all* $\overline{M}_{g,N}$'s. The inductive structure which these maps capture and its use in studying their birational geometry—even if we only ask questions about unpointed curves, our answers will usually involve pointed ones—is a basic theme of these lectures. The simplest of the sets of maps is the set of "forgetful" maps.

Definition 1.19 [Forgetful Maps]: If N is a subset of P, the forgetful map $\pi_{P,N} : \overline{M}_{g,P} \longrightarrow \overline{M}_{g,N}$ is the map which sends the moduli point of an P-pointed stable curve to the moduli point of the N-pointed stable curve obtained by *forgetting* the marked points not indexed by N and taking the stable model of the resulting curve.

The maps $\pi_{(N,q),N} : \overline{M}_{g,(N,q)} \longrightarrow \overline{M}_{g,N}$ is loosely referred to as the universal curve over $\overline{M}_{g,N}$—for example, when $N = \emptyset$, we obtain the universal curve referred to in Definition 1.1. Two warnings are in order here. First, the fiber over a N-pointed curve C is isomorphic to C *only* if C has no automorphisms; in general, it's $C/\mathrm{Aut}(C)$. Even in this case, the curve in $\overline{M}_{g,(N,q)}$ whose moduli point is given by $q \in C$

is only stably equivalent to C when either q is a node of C or $q \in N$. In both cases a blow-up is needed, in the former so that we do not mark a node, in the latter so that the marked points remain distinct. When q is a node, we glue the two points of the normalization of C lying over the node to 0 and ∞ on a copy of $\mathbb{P}^1$ and label the point 1 on $\mathbb{P}^1$ by q. When $q = p_n$ for some $n \in N$, we glue it to the point 0 on a copy of $\mathbb{P}^1$and labelling the points 1 and ∞ on the $\mathbb{P}^1$ by n and q: this process is sometimes called attaching a leg at p_n.

One advantage of this procedure is that it endows the map $\pi = \pi_{(N,q),N}$ with canonical sections $\sigma_n : \overline{M}_{g,N} \longrightarrow \overline{M}_{g,(N,q)}$ indexed by N: $\sigma_n(C)$ is the curve which has a leg attached at p_n. We let Σ_n be the image of σ_n.

You may object that we should really denote Σ_n by something $\Sigma_{n,N}$—using the second subscript to make clear which space the class lives on. We'll occasionally do so but more often we'll rely on the context to make this clear with the view that the simplification which results more than justifies any minor imprecision.

Forgetful maps in turn allow us to extend the definitions of the classes λ and κ to $\overline{M}_{g,N}$ by setting

$$\lambda = c_1(\pi_*(\omega_\pi)) \quad \text{and} \quad \kappa = \pi_*\left(c_1\left(\omega_\pi(\sum_{n\in N} \Sigma_n)\right)^2\right)$$

and to define a new set of classes ψ_n for $n \in N$ by

$$\psi_n = \sigma_n^*(c_1(\omega_\pi)) \ .$$

Note that the class λ is 0 when $g \leq 1$.

The other natural maps we'll need to deal with involve the boundary components of $\overline{M}_{g,N}$. First, what are these? Proposition 1.9 continues to apply so the boundary is again the closure of the locus of stable curves with a single node and this again is the union of loci Δ_{irr} and Δ_i for $i \geq 0$. Since it's usually clear—as here—from the context which $\overline{M}_{g,N}$ is referred to, we usually won't make this explicit in the notation.

There are two differences. First, the locus Δ_{irr} is empty if $g = 0$ by the genus formula. Second, when $N \neq \emptyset$, the Δ_i can be further decomposed. A deformation which preserves the node must also preserve the partition of N into two subsets corresponding to which "side" of the node each marked point lies on. For $0 \leq i \leq g$ and $P \subset N$, we denote by $\Delta_{i,P}$ the locus of curves C with a node which divides C into a component of genus i contain the points indexed by P and a component of genus $g - i$ containing the points indexed by $N \setminus P$. Of course, $\Delta_{i,P} = \Delta_{g-i,N\setminus P}$ and when $i = 0$ [resp: $i = g$] stability implies that we'll only get a non-empty locus if $|P| \geq 2$ [resp: $|N \setminus P| \geq 2$].

The other two sets of basic maps are glueing maps which relate the boundary components of $\overline{M}_{g,N}$ to other moduli spaces of pointed curves.

Definition 1.20 [Glueing Map]: We denote by

1. $\xi : \overline{M}_{g-1,N\cup\{q,r\}} \longrightarrow \overline{M}_{g,N}$ the map, with image the closure of Δ_{irr}, defined by identifying p_q and p_r;
2. $\eta : \overline{M}_{i,P\cup\{q\}} \times \overline{M}_{g-i,(N\setminus P)\cup\{r\}} \longrightarrow \overline{M}_{g,N}$, the map, with image the closure of $\Delta_{i,P}$, again defined by identifying the points p_q and p_r; and,
3. $\theta := \theta_D : \overline{M}_{i,P\cup\{q\}} \longrightarrow \overline{M}_{g,N}$ the map defined by choosing a *fixed* curve D in $\overline{M}_{g-i,(N\setminus P)\cup\{r\}}$ and gluing the point p_r on D to the point p_q on the moving curve in $\overline{M}_{i,P\cup\{q\}}$.

Note that all these maps are finite. In fact, they are all of degree 1 except for the maps ξ which are always of degree 2 (we can swap q and r) and the map η which is of degree 2 in the case where g and $|N|$ are even, $i = \frac{g}{2}$ and $|P| = \frac{|N|}{2}$ and we can swap (i, P) and $(g - i, N \setminus P)$. A first application of these maps is the

Proposition 1.21 *The closures of the loci Δ_{irr} and $\Delta_{i,P}$ are the irreducible components of the boundary Δ of $\overline{M}_{g,N}$.*

which follows directly from the irreducibility of the spaces $\overline{M}_{g,N}$.

But much more is true. To simplify the description, we make the marked points implicit. Every stable pointed curve C determines a labelled dual graph $\Gamma(C)$ with a vertex for each component (labelled with the genus of its normalization), an edge for every node connected the two vertices on whose components the node lies (self-edges are allowed to handle nodes of irreducible type) and a leg or half-edge for each marked point based at the vertex of the component on which the point lies and labelled with the marking.

Exercise 1.22: 1. Two stable pointed curves have isomorphic dual graphs iff they have the same topological type meaning that there is a topological isomorphism between their normalizations preserving the set of nodes and the ordered set of marked points.
2. The set of curves with a given dual graph is a locus of pure codimension equal to the number of edges (i.e. nodes) and these sets form a stratification of $\overline{M}_{g,n}$.

This is usually known as the stratification by topological type. Note that we may view the normalization $\tilde{C}$ of a curve C in $\overline{M}_{g,n}$ as a (possibly disconnected) pointed curve by marking the two preimages of each node. There is then a natural map between the dual graphs by "fusing" the two legs on $\Gamma(\tilde{C})$ to an edge on $\Gamma(C)$. This globalizes and, arguing as above via an induction on the codimension, we see that

Proposition 1.23 *Any stratum of $\overline{M}_{g,n}$ is the image of a product of moduli spaces M_{g_i,n_i} by a finite, proper, surjective glueing map that identifies two marked points to each node. The closure of any stratum of $\overline{M}_{g,n}$ is the image of a product of moduli spaces $\overline{M}_{g_i,n_i}$ by an extension of such a glueing map.*

1.2 Pullback formulae

In order to use the three families of basic maps effectively, we need a dictionary describing how all these classes pull back under them.

This dictionary and various other formulae which follow later are simplified if we think of ψ_n as an "anti-boundary" and define

$$\delta_{0,n} = -\psi_n = \delta_{0,N\setminus\{n\}} \ .$$

It's also useful to introduce the sum of classes:

$$\psi = \sum_{n\in N} \psi_n.$$

We'll also adopt the usual convention that a class $\delta_{i,P}$ on $\overline{M}_{g,N}$ is 0 when $i > g$ or $i < 0$ and when $P \not\subset N$.

Lemma 1.24 *Under the map* $\pi = \pi_{(N,q),N} : \overline{M}_{g,(N,q)} \longrightarrow \overline{M}_{g,N}$,

1. $\pi^*(\lambda) = \lambda$.
2. $\pi^*(\kappa) = \kappa + \delta_{0,\{q\}}$.
3. $\pi^*(\delta_{\text{irr}}) = \delta_{\text{irr}}$.
4. $\pi^*(\delta_{i,P}) = \delta_{i,P} + \delta_{i,(P,q)}$,*w unless* $\delta_{i,P} = \delta_{i,(P,q)}$, *when* $\pi^*(\delta_{i,P}) = \delta_{i,P}$.

Lemma 1.25 *Under the map* $\xi : \overline{M}_{g-1,N\cup\{q,r\}} \longrightarrow \overline{M}_{g,N}$,

1. $\xi^*(\lambda) = \lambda$.
2. $\xi^*(\kappa) = \kappa$.
3. $\xi^*(\delta_{\text{irr}}) = \delta_{\text{irr}} + \sum_{q\in P, r\notin P} \delta_{i,P}$.
4. $\xi^*(\delta_{i,P}) = \delta_{i,P} + \delta_{i-1,(P,q,r)}$.

Lemma 1.26 *Under the map* $\theta : \overline{M}_{i,(P,q)} \longrightarrow \overline{M}_{g,N}$,

1. $\theta^*(\lambda) = \lambda$.
2. $\theta^*(\kappa) = \kappa$.
3. $\theta^*(\delta_{\text{irr}}) = \delta_{\text{irr}}$.
4. $\theta^*(\delta_{i,P}) = \delta_{i,P}$.

Just a few remarks about the proofs. For more details, see [16, Section 2] or [2, Section 1]. The claims for λ follow from the functoriality of the dualizing sheaf. For example, if θ maps $[B]$ to $[C]$ by attaching the fixed curve D then $H^0(C,\omega_C) = H^0(B,\omega_B) \bigoplus H^0(D,\omega_D)$. So $\theta^*(\Lambda_N)$ is the direct sum of $\Lambda_{i,(P,q)}$ and a trivial bundle and the two have the same first Chern class. This argument comes up often enough that we formalize it.

Attachment Lemma 1.27 *For any family $X \longrightarrow B$ of stable curves, let $Y \longrightarrow B$ be the family obtained by attaching a fixed pointed curve (C,p) along a single section of smooth points of fibers and let $\pi : X \longrightarrow Y$ be the inclusion. Then. $\Lambda_Y = \Lambda_X \oplus (H^0(C,w_C) \bigotimes \mathcal{O}_B)$ and hence $\pi^*(\lambda_Y) = \lambda_X$. An analogous result holds if several fixed curves are attached along several disjoint sections, or, if a fixed curve is attached along several sections, or both.*

Likewise Λ pulls back under ξ to the sum of the the Λ in genus $g-1$ and a trivial line bundle (essentially the kernel of the addition map from a sum of trivial bundles at q and r).

The formulae for the boundary classes follow directly by drawing pictures of general curves in each and asking what curves map to them. I'll leave these as exercises. Once these are established the formulae for κ follow immediately from Mumford's Formula 1.44 $\kappa = 12\lambda + \psi - \delta$, discussed in the next section for $N = \emptyset$ but which then follows for all N once 2. of Lemma 1.24 is established. This follows from

Lemma 1.28 *The restriction $\omega(\Sigma)\big|_\Sigma$ is isomorphic, by taking residues, to $\mathcal{O}_\Sigma$. Hence,*

$$\pi_*\left(c_1(\omega)\big|_\Sigma\right) = \pi_*\left(\Sigma\big|_\Sigma\right) = \psi\ .$$

by simply expanding the self-intersection in the definition of κ.

Since $H^1(\overline{M}_{g,n},\mathbb{Q})$ is always 0 (cf. [2]), by Künneth, the second cohomology group of $\overline{M}_{i,P\cup\{q\}} \times \overline{M}_{g-i,(N\setminus P)\cup\{r\}}$ is the direct sum of the the second cohomology groups of the two factors. Thus, the formulae for the maps θ give formulae for the maps ξ by pulling back under the projections onto the two factors and summing.

In our study of nef cones in Chapter 3, we will need a further refinement;

Lemma 1.29 1. *Any line bundle on $\Delta_{i,P}$ of is numerically equivalent to a tensor product of the pushforwards under η of unique line bundles from the two factors $\overline{M}_{i,P\cup\{q\}}$ and $\overline{M}_{g-i,(N\setminus P)\cup\{r\}}$. The given line bundle is nef on $\Delta_{i,P}$ iff each of the line bundles on the factors is nef.*

2. *Dually, let B be any curve on the product and let B'_i and B'_{g-i} be its images on the two factors (with multiplicity given by the pushforward of cycles). This gives curves B_i and B_{g-i} in $\overline{M}_{g,n}$ by gluing on a fixed curve lying the "other" factor and the numerical equivalence classes of these curves depends only on that of B. Conversely, B and $B_i + B_{g-i}$ are numerically equivalent on $\overline{M}_{g,n}$.*

3. *Any curve B in $\overline{M}_{g,n}$ has generic point $[C]$ corresponding to a curve C which decomposes as $F \cup C'$ where the subcurve F is fixed and the subcurve C' traces out a curve B' in some $\overline{M}_{g',n'}$. Any such B is numerically equivalent to one for which the subcurves C' are generically irreducible.*

The first statement follows inductively from the pullback formulae above and the stratification of Proposition 1.23: details may again be found in [16, Section 2] which gives an algorithm for realizing these equivalences explicitly. Once this is established the other assertions are immediate.

There's one more set of classes which come up fairly often. We denote by ω the relative dualing sheaf of the universal curve $\overline{M}_{g,1} \longrightarrow \overline{M}_g$, by $\rho_{i,n}$ the map $\rho_{i,n} : \overline{M}_{g,n} \longrightarrow \overline{M}_{g,1}$ given by forgetting all *but* the i^{th} marked point and define $\omega_i = \rho_{i,n}^*(\omega)$. Warning: ω_i is *not* the relative dualizing sheaf for the map $\pi_{N,N\setminus\{i\}}$ which forgets the i^{th} point (cf. Pointed Canonical Bundle Formula 1.46).

Exercise 1.30: Show that $\omega_p = \psi_p - \sum_{p \in P} \delta_{0,P}$.

Picard groups

The following theorem, for whose proof I'll simply refer to [2, Theorem 2.2], says that we've now seen all the divisor classes on $\overline{M}_{g,n}$.

Theorem 1.31 $\operatorname{Pic}(\overline{M}_{g,N})$ *is generated by the basic classes* λ, κ, δ_{irr}, $\delta_{i,P}$ *and* ψ_n *for all* g *and* N. *For* $g \geq 3$, *the only relations on these classes are:*

1. $\kappa = 12\lambda + \psi - \delta$ *(cf. Mumford's Formula 1.44);*
2. *The symmetries* $\delta_{i,P} = \delta_{g-i,N \setminus P}$.

In smaller genera there are some extra relations. In genus 2, the fact that λ is trivial on M_2 forces it to be a sum of boundaries on $\overline{M}_2$: in Genus 2 λ–Formula 1.52 we'll see that in fact

$$\lambda = \frac{1}{10}\delta_{\text{irr}} + \frac{1}{5}\delta_1 \ ,$$

Exercise 1.32: Use Lemma 1.24 to see that the same relation holds on $\overline{M}_{2,N}$ for any N.

In genus 1, there must again be a relation involving λ. From the description of $\overline{M}_{1,1}$ in the preceding section and the next exercise, we see than it must be $\lambda = \psi = \frac{1}{12}\delta_{\text{irr}}$ on $\overline{M}_{1,1}$. Pulling these relations back we find that

$$\lambda = \frac{1}{12}\delta_{\text{irr}} \quad \text{and} \quad \psi_n = \frac{1}{12}\delta_{\text{irr}} + \sum_{n \in P} \delta_{0,P} \ .$$

for any N.

Exercise 1.33: 1. Show that, if $\rho : X \longrightarrow B$ is a flat family of curves over a smooth complete curve B and with smooth total space X and $\sigma : B \longrightarrow X$ is a section of ρ with image Σ, then $\omega_{X/B} \cdot \Sigma = -\Sigma^2$. Hint: The fiber of $\omega_{X/B}$ at any point is the dual of the tangent space to the

fibers of ρ at that point; at a point of Σ, this is the dual to the fiber of the normal bundle to Σ.
2. Let $\rho : X \longrightarrow \mathbb{P}^1$ be the blowup of a general pencil of cubics at its basepoints (all smooth on each element of the pencil, by genericity) and let Σ be an exceptional divisor of the blowup. Show that X has 12 singular fibers, all pigtails (rational nodal curves), and that $\omega_{X/B} \cdot \omega_{X/B} \cdot \Sigma = -\Sigma^2 = 1$.
3. Since the pencil is generic, B maps non-trivially to $\overline{M}_{1,1}$ hence must be a covering. Deduce the relations $12\lambda = 12\psi = \delta_{\text{irr}}$ on $\overline{M}_{1,1}$.

In genus 0, the locus δ_{irr} is empty (a curve with a non-disconnecting node must have positive genus) as is the class λ (in fact the bundle Λ is zero). In $M_{0,4}$, the three points of the boundary (which correspond to $\delta_{0,\{1,2\}} = \delta_{0,\{3,4\}}$, $\delta_{0,\{1,3\}} = \delta_{0,\{2,4\}}$, $\delta_{0,\{1,4\}} = \delta_{0,\{2,3\}}$) are all linearly equivalent. Pulling these back to $\overline{M}_{0,N}$, we get

Four Point Relations 1.34 *For any subset $Q = \{i, j, k, l\}$ of $\mathbf{n}$ of order 4, the class*

$$\delta_Q := \sum_{i,j \in P, k,l \notin P} \delta_{0,P}$$

depends, as the notation suggests, only on Q and not on choice of the pair of elements i and j.

Keel [40] proves that all relations in genus 0 are consequences of these but it's worth noting a few expressing the classes κ and ψ_n in terms of boundary classes for future reference. The basic case is $\overline{M}_{0,3}$ (a point!) where all these classes are 0.

Exercise 1.35: Show that pulling back these relations from $\overline{M}_{0,3}$ to $\overline{M}_{0,\mathbf{n}}$ gives the relations:

1. $\kappa = \sum_{q,r \notin P} \delta_{0,P}$ for any q and r in $\mathbf{n}$.
2. $\kappa = \sum_{P} \left(\frac{|P|(n-|P|)}{n-1} - 1 \right) \delta_{0,P}$. (Average!)
3. $\psi_n = \sum_{s \in P, q,r, \notin P} \delta_{0,P}$ for any q and r in $\mathbf{n}$ distinct from s.

1.3 Relations amongst divisor classes

A natural goal is to study relations amongst cycle classes on $\overline{M}_{g,n}$ and to deduce geometric properties from this study. A basic difficulty is that most techniques give relations amongst analogous classes on the base of a family $X \longrightarrow B$ of stable curves. If we had a fine moduli space, we'd have some handle on the problem since we could try to work out what these results say for the universal curve but even then we'd have to work over bases of large dimension. What we'd like is a way to work with 1-parameter families of curves and derive conclusions about moduli spaces as in Exercise 1.33.

The standard solution is to enlarging the category of schemes to the category of algebraic stacks which is big enough to contain a stack $\overline{\mathcal{M}}_{g,n}$ representing the functor of families of stable curves. Since. moreover, there is a map (of stacks) $\zeta : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{M}_{g,n}$, we could first prove facts about $\overline{\mathcal{M}}_{g,n}$ and then deduce consequences about $\overline{M}_{g,n}$. The kicker is that the language of stacks is complex and it takes a fair while to learn to work with them. So, here I will simply review the ad hoc or "fat chance" approach used in *Moduli of Curves* [32], referring to Section 3.D for more details.

Likewise, the toolkit for deriving relations on these classes is a fairly hefty one. Here, I have chosen to be even more laconic. Consequences of Grothendieck-Riemann-Roch and Porteous' formula are simply quoted from Section 3.E of *Moduli of Curves* [32] because I will not need others in the sequel. We will have more need for admissible covers but since easily understood schematic diagrams of particular covers will usually suffice I will again simply refer to Section 3.G for details on their global properties. But I will give a sketch of Harris' computation of the class of the hyperelliptic locus in $\overline{M}_3$, both because I will need it (and some corollaries) in later chapters, and because it remains, 20 years after Joe lectured on it at Bowdoin ([30]), the prettiest and most accessible illustration of all the ideas rostered above. Further details, are in Sections 3.F and 3.H of *Moduli of Curves* [32].

Rational divisor classes and formulae for working with them

Definition 1.36: A rational divisor class on the moduli *space* is an element of

$$A^1(\overline{M}_{g,n})\bigotimes\mathbb{Q} = \mathrm{Pic}(\overline{M}_{g,n})\bigotimes\mathbb{Q}.$$

These are equal because, since $\overline{M}_{g,n}$ has only finite quotient singularities, every codimension 1 subvariety of $\overline{M}_{g,n}$ is $\mathbb{Q}$-Cartier.

Definition 1.37: A rational divisor class on the moduli *stack* is an association to each family $\rho : X \longrightarrow B$ of a rational divisor class $\gamma(\rho) \in \mathrm{Pic}(B)\bigotimes\mathbb{Q}$ such that for any fiber square

$$\begin{array}{ccc} X' \cong B' \times_B X & \longrightarrow & X \\ \downarrow{\rho'} & & \downarrow{\rho} \\ B' & \longrightarrow & B \end{array}$$

the class $\gamma(\rho')$ associated to the morphism $\rho' : X' \longrightarrow B'$ is the pullback of the class $\gamma(\rho)$ associated to $\rho : X \longrightarrow B$. The group of rational divisor classes on the moduli stack will be denoted $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_{g,n})\bigotimes\mathbb{Q}$.

Warning If you know just a bit about stacks, you might think this is a definition abstracted from that category. It's not. For example, to specify a line bundle on a stack you need to specify the isomorphism associated to a fiber square. This definition is purely ad hoc: we are not working with stacks. I use the notation $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_{g,n})$ rather than $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ to emphasize this but, for convenience, I will speak abusively of classes in the former and the latter as stack and space classes respectively.

Now that these definitions are behind us, I'll return to omitting the explicit tensor $\mathbb{Q}$'s and just implicitly assume them.

The first fact, which I will simply recall from Proposition 3.88 of *Moduli of Curves* [32] is that these two notions are, in fact, the same. That is, there is a canonical isomorphism $\zeta : \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_{g,n}) \longrightarrow \mathrm{Pic}(\overline{M}_{g,n})$.

In practice, what we need are tools for moving classes from betwen these two groups. A class like the Hodge class λ lives naturally in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_{g,n})$—it's associated to the relative dualizing sheaf. In fact, this is how we get around the non-existence of a universal curve. The boundary classes δ_i (the classes of the divisors Δ_i), on the other hand, live naturally in $\mathrm{Pic}(\overline{M}_{g,n})$. What codimension 1 subvariety do we associate to λ? What stack divisor class do we associate to Δ for a family of curves *all* of which are singular?

To begin with, let's note several standard reductions.

1. Any class γ in $\mathrm{Pic}(\overline{M}_{g,n})$ is determined by its values on families $\rho : X \longrightarrow B$ with smooth, one-dimensional base B (since two line bundles which agree on every smooth curve are equal).
2. Since we know that $\mathrm{Pic}(\overline{M}_{g,n})$ is discrete, these values are determined by degrees $\deg(\gamma(\rho)) \in \mathbb{Q}$ for such families ρ.
3. Both the reductions above apply with the additional restriction that $\varphi(B)$ can be taken not to lie inside any proper subvariety of $\overline{M}_{g,n}$.

In other words, another way to think of a rational divisor class on the moduli stack is as something that measures the non-triviality of a one-parameter family—we'll stick to these henceforth—by counting the number of fibers of some type (e.g., δ counts the number of singular fibers). This will let us avoid having to associate loci in $\overline{M}_{g,n}$ to classes like λ that are naturally defined for families and we'll only need one notation for both classes.

We can and must turn this around for classes defined as geometric loci in $\overline{M}_{g,n}$. To any closed codimension 1 subvariety $\Sigma \subset \overline{M}_{g,n}$ we can associate a rational divisor class σ by associating a number to each family $\rho : X \longrightarrow B$, naively "the number of elements of the family lying in Σ". To do this, we fix a small neighborhood of $b \in B$ which maps to the versal deformation of X_b giving us a diagram:

$$\begin{array}{ccc} b \in U & \xrightarrow{\tilde{\varphi}} & \mathrm{Def}(X_b) \supset \tilde{\Sigma} \\ \downarrow & & \downarrow \psi \\ B & \xrightarrow{\varphi} & \overline{M}_{g,n} \quad \supset \Sigma \end{array}$$

There are two cases shown on the left and right of the figure below:

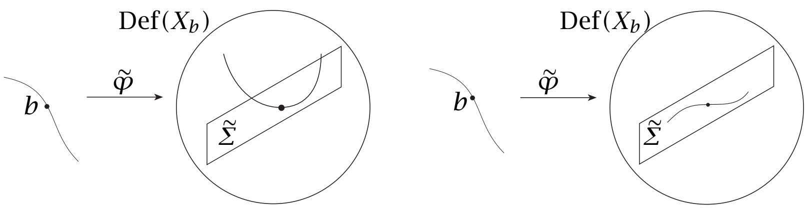


FIGURE 1.38: Counting fibers: the two cases

1. *Only finitely many fibers X_b lie in Σ:* Here we want to assign each such fiber a multiplicity and sum up. We do this using the universal deformation $\mathrm{Def}(X_b)$ space of X_b. On the one hand, Σ determines a divisor $\tilde{\Sigma}$ in this space (Cartier, since $\mathrm{Def}(X_b)$ is smooth). On the other hand, a small neighborhood of $b \in B$ maps to $\mathrm{Def}(X_b)$ and we set $\mathrm{mult}_b(\sigma)$ to be the multiplicity of the pullback of $\tilde{\Sigma}$ under this map. This is well-defined because two such maps agree on a possibly smaller neighborhood.
2. *All fibers X_b lie in Σ:* Formally, we could avoid this case (cf., the reductions above) but it's actually easier to tackle it directly. The idea is to describe $\sigma(\rho)$ as a line bundle L on B: for L, we take the pullback of the normal bundle to $\tilde{\Sigma}$ in $\mathrm{Def}(X_b)$. This is well-defined because $\mathrm{Def}(X_b)$ is also a universal deformation of fibers near b by "openness of versality".

This raises the question: what is the relation between the class σ just defined and the one given by applying "ζ^*" to the rational divisor class $[\Sigma] \in \mathrm{Pic}(\overline{M}_{g,n})$? The answer again comes from deformation theory. If the general member C of a divisor Σ has automorphism group of order n, then the map $\psi : \mathrm{Def}(X_b) \longrightarrow \overline{M}_{g,n}$ will be ramified to order n along Σ (essentially because in the definition of a deformation automorphisms are rigidified away) so:

Proposition 1.39 *If the automorphism group of a general point $[C]$ of Σ has order n, then $\sigma = \frac{1}{n}\zeta^*[\Sigma]$.*

In other words, for irreducible divisors, the two versions agree "up to case" except in genus 2, for the hyperelliptic locus H in genus 3 and for $\Delta_{1,\emptyset}$ for $g \geq 2$. The upshot for finding relations between classes is expressed in:

Basic Dictionary 1.40 *Fix corresponding classes $\Gamma \in \mathrm{Pic}(\overline{M}_{g,n})$ and $\gamma \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$: i.e., $\gamma = \zeta^*(\Gamma)$. Let $\Sigma_1, \ldots, \Sigma_k$ be irreducible codimension 1 subvarieties of $\overline{M}_{g,n}$, let $\sigma_1, \ldots, \sigma_k$ be the classes they determine in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ and let a_i be the order of the automorphism group of a general member of Σ_i. Then the following statements are equivalent:*

1. *The relation $\sum_i c_i \cdot \sigma_i(\rho) = \gamma(\rho)$ holds in $\mathrm{Pic}(B)$ for every one-parameter family $\rho : X \longrightarrow B$ of stable curves of genus g.*
2. *The relation $\sum_i \left(\frac{c_i}{a_i}\right) \cdot [\Sigma_i] = \Gamma$ holds in $\mathrm{Pic}(\overline{M}_{g,n}) \bigotimes \mathbb{Q}$.*

Moreover, the second statement follows if we know the first for families whose general fiber does not lie in any of the Σ_i.

Remark 1.41: We can actually work with families $\rho : X \longrightarrow B$ which are only generically stable: to define the degree of γ on such a family make a semi-stable reduction $\rho' : X' \longrightarrow B'$ and divide the degree of γ on the new family by the order of the base change $B' \longrightarrow B$. In this sense, the first statement above follows for such families if the second is known.

Finally, note that because Δ contains the component $\Delta_{1,\emptyset}$ whose generic element has automorphism group of order 2, the divisor class δ on the moduli stack does *not* agree with $\zeta^*(\Delta)$. Rather, $\zeta^*(\Delta) = \delta + \delta_{1,\emptyset}$.

To get a more concrete feel for what is involved and because we'll use the answer extensively later, let's work out the most basic example and determine the rational divisor class δ on the moduli stack associated to the codimension 1 subvariety Δ. To do this, fix a family $\rho : X \longrightarrow B$ with smooth one-dimensional base B and a local parameter t on B.

To begin with let's suppose that the general fiber X_t is smooth and that the special fiber X_0 has a single node at p. We can then choose coordinates x and y on X near p so that $xy = t^k$ for some k. If so, then in the versal deformation space of the nodal curve X_0, the image of B will be a curve with contact of order k with the (smooth) hypersurface of singular deformations. The germ of the image of B in $\overline{M}_{g,n}$ near 0 will be tangent to Δ to order k hence, the defining equation of $\Delta \subset \overline{M}_{g,n}$ will have pullback to B vanishing to order exactly k at $t = 0$. This form generalizes straightforwardly to the case where X_0 has n nodes p_i with local defining equations $xy = t^{k_i}$:

$$\mathrm{mult}_0(\delta) = \sum_{i=1}^{n} k_i \, .$$

What if the general fiber is singular—say, to keep things simple, there is a single node p_b in each fiber X_b? Now we want to use the main claim of Proposition 1.9, that the normal bundle to the discriminant hypersurface in the versal deformation space is isomorphic to the tensor product of the tangent spaces to the two branches of X_b at p_b. To do so, first let $\nu : \widetilde{X} \longrightarrow X$ be the normalization of the total space X followed by a base change if needed, so that the inverse images of the nodes, $\widetilde{\Gamma}$, consists of two disjoint sections $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$. The map $\widetilde{\rho} := \rho \circ \nu : \widetilde{X} \longrightarrow B$ is smooth so the description above translates to say that

$$\delta(\rho) = N_{\widetilde{\Gamma}_1/\widetilde{X}} \bigotimes N_{\widetilde{\Gamma}_2/\widetilde{X}} .$$

which we interpret as an equality in Pic(B) by making the canonical identifications of the $\widetilde{\Gamma}_i$'s with B.

Finally, we can pass to the case where the general fiber has any number of nodes and combine this with the case where there are extra nodes on the special fiber to arrive at:

Lemma 1.42 [Description of δ] *Let $\rho : X \longrightarrow B$ be a family of stable curves of genus g over a smooth, one-dimensional base B whose general fiber has n nodes. Let $\widetilde{X} \longrightarrow X$ be the normalization of the total space of X and $\widetilde{\rho} : \widetilde{X} \longrightarrow B$ the composition. Let $\Gamma \subset X$ be the positive-dimensional components of the singular locus of ρ, and suppose, by making a base change if necessary, that $\widetilde{\Gamma} \subset \widetilde{X}$ the inverse image of Γ in $\widetilde{X}$ consists of $2n$ disjoint sections $\widetilde{\Gamma}_i$. For each point p in the singular locus sing$(\widetilde{\rho})$ of the map $\widetilde{\rho}$, let $k(p)$ be the unique integer such that there exist local coordinates x, y, t on $\widetilde{X}$ near p satisfying $xy = t^{k(p)}$. Then*

$$\delta(\rho) = \bigotimes_{i=1}^{2n} N_{\widetilde{\Gamma}_i/\widetilde{X}} \otimes \mathcal{O}_B\Big(\sum_{p \in \mathrm{Sing}(\widetilde{\rho})} k(p) \cdot \widetilde{\rho}(p) \Big).$$

In particular, the degree of δ is given by

$$\deg(\delta(\rho)) = (\widetilde{\Gamma})^2 + \sum_{p \in \mathrm{Sing}(\widetilde{\rho})} k(p).$$

Remark 1.43: The freedom to pass to semi-stable families noted in Remark 1.41 is often useful where we'd like to work with family having smooth total space X. If $k(p) = n$ above, then X will have an A_{n-1} singularity at p, but we can resolve this singularity by $(n-1)$ blowups replacing the point p by a chain of $(n-1)$ rational curves on which there are n nodes q_i, each with $k(q_i) = 1$.

We leave it as an exercise to the reader to formulate variants of Lemma 1.42 describing the classes of the individual components $\delta_{i,P}$

by specifying the types of the nodes and the locations of the marked points.

Finally, I want to quote a few basic relations amongst divisor classes that follow from the Grothendieck-Riemann-Roch formula and Porteous' formula. I won't even recall these tools here except to note that they naturally give relations amongst stack classes and that it's in applying these tools that some version of the formalism above (or an honest study of the stacky version we are trying to avoid) becomes essential.

We'll need two formulae proved using Grothendieck-Riemann-Roch (and I'll also use it at a couple of points in Section 3.1). The first, Mumford's Formula 1.44, relates the classes κ and λ. For $\overline{M}_g$, this was originally proved by Mumford's [53] and his argument may also be found in Section 3.E of *Moduli of Curves* [32]. The proof for $\overline{M}_{g,n}$ is a fairly straightforward generalization. or can be deduced from the unpointed case using Lemma 1.24.

Mumford's Formula 1.44 $\kappa = 12\lambda + \psi - \delta$

The Canonical Bundle Formula 1.45, which we will only need in the unpointed case proved in Section 3.E of *Moduli of Curves* [32], calls for a bit of comment, since $\overline{M}_g$ is singular. is we need to fiddle a bit to define this class. There's no problem defining a canonical bundle on the smooth locus of $\overline{M}_g$: we just take the bundle generated by holomorphic differential forms of (top) degree $(3g-3)$. But this locus has codimension at least 2 for $g \geq 3$ (cf. Exercise 2.27 of *Moduli of Curves* [32] and recall that points of a divisor, like δ_1 or the hyperelliptic locus in genus 3, whose generic member carries a single involution are, in fact, smooth) so there is a unique rational line bundle extending the canonical bundle on the smooth locus. Its stack class is given by,

Canonical Bundle Formula 1.45 $K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta$

In terms of classes on $\overline{M}_g$ this last becomes

$$\begin{aligned} K_{\overline{M}_g} &= 13\lambda - 2[\Delta_{\mathrm{irr}}] - \frac{3}{2}[\Delta_1] - 2[\Delta_2] - \cdots \\ &= 13\lambda - 2\delta - \delta_1 . \end{aligned}$$

Since I am quoting, I'll state the pointed generalization: see [49, Theorem 2.6] for a proof in terms of the classes ω_i.

Pointed Canonical Bundle Formula 1.46

$$K_{\overline{M}_{g,n}} = 13\lambda + \psi - 2\delta - \sum_P \delta_{1,P}$$

Test curves

Often we are given a geometrically defined codimension 1 subvariety E of $\overline{M}_{g,N}$ and we'd like to express the associated stack class e as a linear combination of basic divisor classes. Tools like the Grothendieck-Riemann-Roch formula provide no help because we cannot describe the corresponding line bundle (particularly its Chern class) explicitly. In such cases, the most effective method is the method of *test curves*.

This begins by writing down e as a linear combination of basic classes with undetermined coefficients. Let's, for simplicity, stick to $\overline{M}_g$ where this would mean writing

$$e = a\lambda + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{\lfloor g/2 \rfloor} b_i \delta_i .$$

Given any one-parameter family $\rho : C \longrightarrow B$ of stable curves of genus g this relation implies the relation

$$\deg(e(\rho)) = a \deg(\lambda(\rho)) + b_{\mathrm{irr}} \deg(\delta_{\mathrm{irr}}(\rho)) + \sum_{i=1}^{\lfloor g/2 \rfloor} b_i \deg(\delta_i(\rho)) .$$

amongst the degrees of the corresponding divisor classes on the base B. The idea of the method is to turn this around by calculating all these degrees and interpreting this equation as giving a relation on the undetermined coefficients. If we can find enough families for which these relations are independent, we can solve for the undetermined coefficients.

At first, we might hope to work with families of smooth curves. There are two objections: first, as g increases it becomes harder and harder to write down families for which the degrees $\deg(e(\rho))$ and $\deg(\lambda(\rho))$ are accessible. This is a reflection of the fact, explored in the next lecture, that $\overline{M}_g$ becomes less like a rational variety and more like a variety of general type as g increases. Moreover, such families, by definition, have $\deg(\delta_{\text{irr}}(\rho)) = 0$ and $\deg(\delta_i(\rho)) = 0$ for all i so they *all* give the same relation letting us solve for a but giving no information on the b_i's. Next, we could hope to to work with generically smooth families. Here again it soon becomes hard to write down families where the degrees are easy to compute and the the degrees of the higher δ_i's are non-zero. The solution is to work with families consisting *entirely* of singular curves.

I want to give three simple examples and then use them to deduce a relation we'll need later.

Example 1.47: Fix a curve D of genus $(g-1)$ and an elliptic curve E and attach a fixed point p of E to a varying point q of D. In other words, the total space X of our family would be the disjoint union of $D \times D$ and $D \times E$ modulo the identification of the diagonal Δ of $D \times D$ with $D \times \{p\}$ in E as shown below. This family lies inside Δ_1.

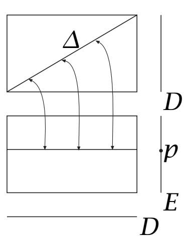


Figure 1.48: Moving point on D attached to fixed point on E

EXAMPLE 1.49: Fix a curve D of genus $(g-1)$ and identify a fixed point p of D with a varying point q of D. This gives a family lying in $\Delta_{\rm irr}$. However, when we take the stable reduction of this family, the fiber over p itself (that is, where q approaches p) is a copy of D joined by a disconnecting node to a rational curve with a node or "pigtail" and this family therefore meets Δ_1 once. (To see this, begin with $D \times D$ as in the diagram on the left, then blowup the point (p,p) obtaining the diagram on the right and finally identify the now disjoint sections $D \times \{p\}$ and Δ to get the bottom diagram.) By the uniqueness of stable reductions, this special fiber *must* give the limit in $\overline{M}_g$ as $q \longrightarrow p$.

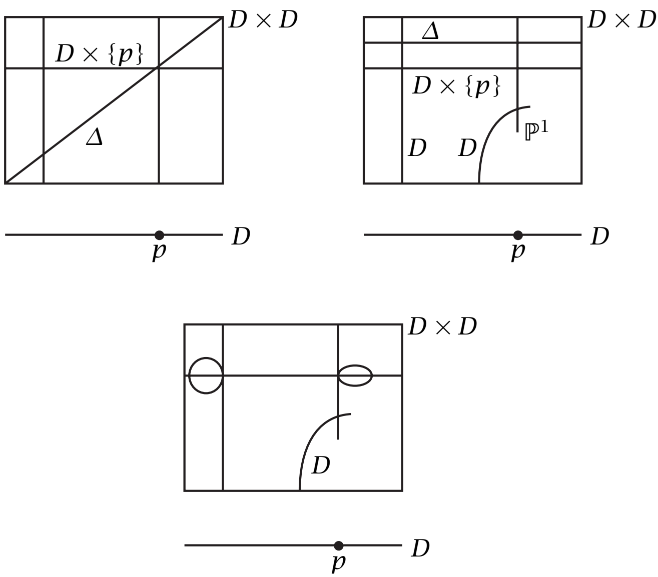


FIGURE 1.50: Moving point on D attached to fixed point on D

EXAMPLE 1.51: Fix a curve D of genus $(g-1)$ and identify a fixed point p of D with a base point q of a generic pencil of plane cubic curves E to obtain a family of stable curves of genus g over $\mathbb{P}^1$. As the elliptic curves degenerate, we again pick up a special fibers with a "pigtail", or rational nodal curve.

I've recorded in the table below, the intersection numbers of each with the standard classes. I haven't listed the degrees of the δ_i's for $i \geq 2$ because these are all clearly 0.

	Example 1.47	Example 1.49	Example 1.51
$\deg(\lambda)$	0	0	1
$\deg(\delta_{\text{irr}})$	0	$2-2g$	12
$\deg(\delta_1)$	$4-2g$	1	-1

I'll verify the first two columns of the table and leave the third as an exercise. To begin with, lets look at Example 1.47 in which the fiber X_q is the union of D and E with q in D identified to p in E. By the Attachment Lemma 1.27, $\deg_D(\lambda) = 0$. The degree $\deg_D(\delta_{\text{irr}})$ is also 0 because each fiber X_q contains a single disconnecting node. For the same reason, the image of this family in moduli lies entirely in Δ_1. To find $D \cdot \Delta_1$, therefore, we apply our calculation of δ , which says that the value on D of the divisor class δ (or, equivalently in this case, δ_1) on the moduli stack is the tensor product of the normal bundles $N_{D\times\{p\}/D\times E} \bigotimes N_{\Delta/D\times D}$. The first factor here is trivial, and the second has degree equal to the self-intersection of the diagonal Δ in the product $D \times D$ of a curve of genus $g-1$ with itself. This is just the topological Euler characteristic of D, which is $1-2(g-1)+1 = 4-2g$. Since test curves often lie in a component of Δ, this type of normal sheaf argument occurs frequently.

Example 1.49 illustrates this in a somewhat dual manner. Only the fiber X_p contains a disconnecting node and since the surface X is smooth at this point it follows that $\deg_D(\delta_1) = 1$. However the image of this family in moduli lies entirely in Δ_{irr}, so we again need to compute the restriction to D of the normal bundle to Δ_{irr} in $\overline{M}_g$ to evaluate $\deg_D(\delta_{\mathrm{irr}})$. Here this bundle is the tensor product of the normal bundles to the proper transforms of Δ and of $D \times \{p\}$ on the blowup of $D \times D$ at (p, p). On $D \times D$, $\Delta^2 = 4 - 2g$ and $(D \times \{p\})^2 = 0$. Since each curve passes through (p, p), each self-intersection drops by one when we blow up, yielding $\deg_D(\delta_{\mathrm{irr}}) = 2 - 2g$.

To calculate $\deg_D(\lambda)$, we use the exact sequence on X_q

$$0 \longrightarrow H^0(K_D) \longrightarrow H^0(\omega_{X_q}) \xrightarrow{\mathrm{res}_p} \mathbb{C} \longrightarrow 0.$$

The corresponding sequence of direct images is

$$0 \longrightarrow H^0(K_D) \bigotimes \mathcal{O} \longrightarrow \pi_*(\omega_{X/D}) \longrightarrow \mathcal{O} \longrightarrow 0$$

from which it's immediate that the first Chern class of $\pi_*(\omega_{X/D})$ is trivial and, hence, that $\deg_D(\lambda) = 0$.

We leave Example 1.51 as a complement to Exercise 1.33.

As a first application, we consider the case $g = 2$. What is special here is that $\mathrm{Pic}(M_2) = 0$. (This can be seen by recalling that a smooth curve of genus 2 is determined by its Weierstrass points: so M_2 can be expressed as a quotient of the affine variety $(\mathbb{P}^1)^6$ minus all diagonals by the action of the symmetric group of order six.) Thus, the class $\lambda \in \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_2) \bigotimes \mathbb{Q}$ must be expressible as a linear combination of the boundary classes δ_{irr} and δ_1. If we write $\lambda = b_{\mathrm{irr}}\delta_{\mathrm{irr}} + b_1\delta_1$ then the last two columns in the table above give the relations $0 = -2b_{\mathrm{irr}} + b_1$ and $1 = 12b_{\mathrm{irr}} - b_1$ which solve to give the "extra" relation in $\mathrm{Pic}(\overline{M}_2) = 0$ which we quoted above.

Genus 2 λ–Formula 1.52

$$\lambda = \frac{1}{10}\delta_{\mathrm{irr}} + \frac{1}{5}\delta_1 .$$

More typical applications of test curves are to determining the classes of loci of curves possessing special linear series. A model problem is the determination of the the stack class $\overline{h}$ of the divisor $\overline{H}$ in $\overline{M}_3$ obtained by taking the closure of the hyperelliptic locus in M_3. To get such a relation, however, it is necessary to understand when a singular stable curve lies in $\overline{h}$. A very workable answer to this, and similar questions, is provided by the theory of admissible covers discussed in Section 3.G of *Moduli of Curves* [32]. For the families above, however, we can guess the answer from the principle that fibers in $\overline{H}$ must carry a "hyperelliptic" involution which fixes the nodes (but not any component)—and blind faith that the intersections in question are transverse (or Exercise 1.53).

In Example 1.47, this happens for the six fibers for which the point q on the genus 2 curve D is a Weierstrass point: thus $\deg(\overline{h}(\rho)) = 6$. In Example 1.49, there is one such fiber. It's *not* the one with the pigtail since the pigtail has no involution; rather it's the fiber in which the moving point q is the image of the fixed point p under the hyperelliptic involution on the genus two curve D. In Example 1.51, there are no such fibers since the point of attachment of the elliptic curve to the genus 2 curve D is general on D (in particular, *not* a Weierstrass point) and any such involution would have to ramify at D.

Thus, if we write $\overline{h} = a\lambda + b_{\text{irr}}\delta_{\text{irr}} + b_1\delta_1$ we have

$$\begin{aligned} 6 &= 0a + \ 0b_{\text{irr}} - 2b_1 \\ 1 &= 0a + \ 4b_{\text{irr}} + 1b_1 \\ 0 &= 1a + 12b_{\text{irr}} - 1b_1 \end{aligned}$$

which solves to give

$$\overline{h} = 9\lambda - \delta_{\text{irr}} - 3\delta_1 \,.$$

Taking account of the fact that generic curves in both $\overline{H}$ and Δ_1 carry an involution this gives the relation

$$\overline{H} = 18\lambda - 2\delta_{\text{irr}} - 3\delta_1$$

in $\text{Pic}(\overline{M}_3)\bigotimes\mathbb{Q}$.

Exercise 1.53: An application of Porteous' theorem gives the stack class of the hyperelliptic locus h in $\mathcal{M}_3$ as 9λ (cf. pp. 162-4 of *Moduli* *of Curves* [32]). Since the general curve in the corresponding locus H in M_3 has exactly 2 automorphisms, this means that $H = 18\lambda$. Use this and the fact that the class $\overline{h}$ has degree 0 on the curve of Example 1.51 to deduce that the coefficient a above must equal 9 and conclude that the intersections of the other two curves with $\overline{H}$ are, in fact, transverse.

Chapter 2

Cones of effective divisors

The theme of this lecture is the study of cones of effective divisors and we will be principally concerned with $\mathrm{NE}^1(\overline{M}_g)$ until the very end. Results in this area are of two types that I will refer to loosely as upper and lower bounds. More precisely,

Definition 2.1: Given two cones L and U in the same real vector space V such that $L \subset U$ we say that L is a lower bound for U and U is an upper bound for L.

For example, since the sum of an very ample class and an effective class is big and effective, we see that the sum of $\mathrm{Amp}(\overline{M}_g)$ and an effective ray gives a lower bound for $\mathrm{NE}^1(\overline{M}_g)$. This means that a lower bound can be produced by computing the coordinates of any particular effective divisor in terms of the natural basis of $\mathrm{Pic}(\overline{M}_g)$ given by λ and the boundary classes and applying estimates for the ample cone that we'll cover in Chapter 3. Carrying this out will be the theme of the first two sections.

Upper bounds on $\mathrm{NE}^1(\overline{M}_g)$ are most readily obtained by finding test curves B whose deformations fill out $\overline{M}_g$ and computing the degrees of λ and the boundary classes on B. The

Effective Dichotomy 2.2 *If B is an effective curve in $\overline{M}_g$ and D is an effective divisor, then either* $\deg_D(B) \geq 0$ *or* $B \subset D$.

says that an effective divisor must have non-negative degree on such a B, so each such curve yields an test *in*equality satisfied by the coefficients of effective divisors. The third section works this out for some examples due of Harris and the author.

Currently, the gap between the known upper and lower bounds is, with few exceptions, more of a chasm. So I could not resist breaking my general rule against introducing spaces of stable maps to briefly discuss in the final section a recent theorem proved independently by Coskun, Harris and Starr [10] and Keel [42] which relates the effective cone of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, d)$ to that of $\overline{\mathcal{M}}_{0,n}$. This in turn has been computed for $n \leq 5$ by Keel and McKernan [43].

2.1 The Brill-Noether Ray Theorem

Setup for and statement of the theorem

To begin with, I'd like to present the classical lower bound obtained by computing the coordinates of what are called Brill-Noether divisors. This result, due to Harris and Mumford [33] inaugurated the subject of this lecture. The lower bound it gives is, 25 years later, still nearly the best known: recent work of Farkas discussed in Section 2.2 gives slightly better bounds for many g. The proof we'll outline a an elegant streamlining of the original argument due to Eisenbud and Harris [14].

We should establish two points of notation first. The Brill-Noether Ray Theorem is a calculation in the Picard group of the moduli space $\overline{M}_g$ and the divisors we'll be considering will come to us as subschemes $D \subset \overline{M}_g$ of the moduli space. However, we'll to carry out the necessary calculations in the group $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ of rational divisor classes on the moduli stack and will abuse notation by using the same letter D

to denote an effective divisor $D \subset \overline{M}_g$ and the counting class in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ associated to it. (Recall that this counting class coincides with the the class $\zeta^*([D]) \in \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ associated to $[D] \in \mathrm{Pic}(\overline{M}_g)$ given by isomorphism of stack and space Picard groups except in genus 2 or when the support of D contains the divisor Δ_1 in general or the divisor $H_3 \subset \overline{M}_3$ of hyperelliptic curves of genus 3. None of these exceptions will be relevant here.)

We will coordinatize a divisor class

$$D = a\lambda - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i \delta_i \,.$$

The minus signs are a deliberate (and fairly standard) departure from the notation of Chapter 1 designed to yield positive b-coefficients. We will refer to the ratio $s_D := \frac{a}{b_{\mathrm{irr}}}$ as the *slope* of D. In the sequel, it is universally the case that a divisor of slope s can be written in the form $c(s\lambda - \delta) + \sum_{i=1}^{\lfloor g/2 \rfloor} c_i \delta_i$ with both c and all the c_i positive so we will abuse language slightly and *assume* this without comment when we speak of a divisor of slope s.

Warning: this convention for defining slopes has the unfortunate consequence that constructing effective divisors of *small* slope gives good *lower* bounds for $\mathrm{NE}^1(\overline{M}_g)$ while proving they cannot exist gives *upper* bounds.

With all this said, we can define the divisors we want to study.

Definition 2.3 [Brill-Noether Divisor]: Loosely, a Brill-Noether divisor is the locus of curves that carry a g^r_d—with r and d fixed—for which the Brill-Noether number

$$\rho = g - (r+1)(g - d + r)$$

is equal to -1. More carefully, a Brill-Noether divisor is the union of the codimension 1 components of the closure of the locus of smooth curves possessing such a linear series.

One defect these divisors have is that they exist only for certain g. Since we're assuming that $\rho = -1$, $g+1$ must be composite and our bounds apply only in this case. For other g, Brill-Noether divisors can be replaced by certain *Petri divisors* which we won't even define here but the bounds obtained are slightly weaker and the computations become *much* more complicated. We will loosely refer to loci of curves possessing exceptional linear series as loci of "special" curves.

We can rewrite the condition $\rho = -1$ in terms of r and the projective dimension $s = g - d + r - 1$ of the linear series residual to the given one in the canonical series as

$$g = (r+1)(s+1) - 1\,.$$

Under this assumption, d, r and s are also related by

$$d = r(s+2) - 1\,.$$

Of course, in view of these constraints, once g is fixed any of the quantities r, d and s determines the other two. However, it'll simplify statements of several propositions to index these divisors by both r and s. We will thus define $D^r_s \subset \overline{M}_g$ to be union of the codimension 1 components of the closure of the locus of smooth curves possessing such a g^r_d.

The result we're after is the calculation of D^r_s, up to a positive rational multiple, in terms of basic classes.

Brill-Noether Ray Theorem 2.4 *Whenever* $s \geq 3$, $r \geq 1$ *and* $g = (r+1)(s+1) - 1$, *the class of* D^r_s *on* $\overline{M}_g$ *is given, for some rational number* $c > 0$, *by*

$$D^r_s = c\Big(\big(g+3\big)\lambda - \Big(\frac{g+1}{6}\Big)\delta_{\mathrm{irr}} - \sum_{i=1}^{\lfloor g/2 \rfloor}\big(i(g-i)\big)\delta_i\Big).$$

Note the remarkable fact that the coefficients (apart from c) depend only on g, not on r or s. In other words, when g factors in several ways the corresponding Brill-Noether divisors all lie on the same *ray* in $\mathrm{Pic}(\overline{M}_g)$.

The Harris-Mumford Theorem

Before turning to the proof of the Brill-Noether Ray Theorem 2.4, we give a celebrated corollary.

Harris-Mumford Theorem 2.5 *The moduli space of curves of genus g is of general type if $g \geq 24$.*

Of course, this "corollary" was what Harris and Mumford were really after. I've reversed the historic roles and cast the Brill-Noether Ray Theorem 2.4 as the star not just to match the theme of these notes but because its really the main step in the argument. The Harris-Mumford Theorem marked a watershed in the study of moduli spaces of curves, especially the kind of questioned considered here. It encouraged, by example, the study of whole range of subtler and deeper questions.

The Harris-Mumford theorem follows from the Brill-Noether Ray Theorem by a criterion that relates the Kodaira dimension of $\overline{M}_g$ to the existence of certain effective divisors $D \subset \overline{M}_g$.

General Type Criterion 2.6 *$\overline{M}_g$ is of general type if there exists an effective divisor D, as above, with*

$$\frac{a}{b_{\text{irr}}} < \frac{13}{2}, \quad \frac{a}{b_i} < \frac{13}{2} \quad \textit{for all } i\textit{, and} \quad \frac{a}{b_1} < \frac{13}{3}.$$

This criterion condenses (in our abusive language) to the assertion that there is an effective D of slope less than $\frac{13}{2}$. A little arithmetic with the coefficients in the Brill-Noether Ray Theorem 2.4 will quickly convince you that the only $\frac{a}{b}$-ratio in the expression for D^r_s that is substantially larger than 1for large g is

$$\frac{a}{b_{\text{irr}}} = \frac{g+3}{\left(\frac{g+1}{6}\right)} = 6 + \frac{12}{g+1}.$$

This is less than $\frac{13}{2}$ for $g \geq 24$ so the Harris-Mumford Theorem 2.5 follows by applying the first part of the criterion for such g, when $g+1$ is composite. When $g+1$ is prime, the same conclusion follows using calculations of the classes of Petri divisors.

The General Type Criterion 2.6 itself follows almost immediately from two facts. The first is the computation of the canonical class of $\overline{M}_g$ in the Canonical Bundle Formula 1.45:

$$K_{\overline{M}_g} = 13\lambda - 2[\Delta_{\mathrm{irr}}] - \frac{3}{2}[\Delta_1] - 2[\Delta_2] - \cdots = 13\lambda - 2\delta_{\mathrm{irr}} - 3\delta_1 - 2\delta_2 - \cdots .$$

The second, immediate from the Cornalba-Harris theorem which will be proved in the next lecture (cf. Cornalba-Harris Theorem 3.9), is the ampleness of $\kappa = 12\lambda - \delta$ (or any divisor $a\lambda - b\delta$ with $\frac{a}{b} > \frac{13}{2}$). Together these two facts show that if there is an effective divisor D as in the criterion, then for suitably divisible m we can find an effective divisor E and a very ample divisor H such that

$$K_{\overline{M}_g}^{\otimes m} = H + E .$$

In particular, this shows that the Hilbert function

$$h^0(\overline{M}_g, K_{\overline{M}_g}^{\otimes mn})$$

has order in n at least that of the divisor $H^{\otimes n}$: this is just another way to say that this order is maximal, or, equivalently, that $K_{\overline{M}_g}$ is big, or, in turn, that $\overline{M}_g$ is of general type.

There is one point in this argument which needs to be addressed. As we remarked when stating the Canonical Bundle Formula 1.45, $\overline{M}_g$ doesn't have a canonical bundle per se. We simply defined $K_{\overline{M}_g}$ to be the unique (rational) line bundle on $\overline{M}_g$ extending the canonical bundle on its smooth locus. There is thus no guarantee that a global regular section of a power of $K_{\overline{M}_g}$ will yield a pluricanonical form on a desingularization of $\overline{M}_g$. In order to ensure that this is in fact the case, we need to study more closely the singularities of $\overline{M}_g$. What must be checked is what was stated classically as the property that "the singularities of $\overline{M}_g$ don't impose adjunction conditions", or, in the language of contemporary birational geometry, that "$\overline{M}_g$ has only *terminal singularities*". Fortunately, the Reid-Tai criterion provides a very effective method of checking whether any finite quotient singularity—and recall that all singularities of $\overline{M}_g$ are of this type—is terminal. We will give no details here and simply refer to Mumford's argument in the original Harris-Mumford paper [33]. You should be aware, however, that this verification involves some lengthy and nontrivial combinatorial complications, since, for each g, we find a different menagerie of such singularities on $\overline{M}_g$. Indeed, the last step in the argument requires a computer verification whose BASIC program listing must surely be the only one ever to appear in *Inventiones*!

Before turning back to the proof of the Brill-Noether Ray Theorem 2.4, I'll simply note that Adam Logan has [49] proved general type results for spaces of pointed curves by a similar strategy.

Theorem 2.7 *For every $g \geq 4$, there is an $n_0(g)$ such that for all $n \geq n_0(g)$, $\overline{M}_{g,n}$ is of general type. For $g = 2$ and $g = 3$, there is an $n_0(g)$ such that for all $n \geq n_0(g)$, $\overline{M}_{g,n}$ dominates a variety of general type.*

Since a variety which maps dominantly to a variety of general type with fibers of general type is again of general type, all that's required is to find $n_0(g)$ such that $\overline{M}_{g,n_0}$ is of general type and the Harris-Mumford theorem takes care of all $g \geq 24$ with $n_0(g) = 0$. Logan finds suitable $n_0(g)$ for g in the range from 4 to 23: in some cases, Logan showed his values to be minimal but recent work of Farkas discussed in the next section shows that not all are.

Pullbacks of Brill-Noether divisors

Eisenbud and Harris prove the Brill-Noether Ray Theorem 2.4 via a sort of wholesale version of the method of test curves. Every time we find a family $X \longrightarrow B$ over a curve B such that D is disjoint from the image of B in $\overline{M}_g$, we get a relation on the coefficients a and b_i. Morally speaking, if we can do the same thing with B a variety of higher dimension, we ought to get several relations. This is essentially what Eisenbud and Harris do. The B's they use are the images of certain smaller moduli spaces under θ maps like those in Definition 1.20. Their plan is to study the pullbacks of D^r_s to these smaller spaces, show that these pullbacks lie in certain special subloci and then show that the coefficients of divisors whose pullbacks lie in these subloci satisfy various relations which together yield the Brill-Noether Ray theorem.

The first space they use is the moduli space $\overline{M}_{0,g}$ of stable g-pointed rational curves equipped with the map $u : \overline{M}_{0,g} \longrightarrow \overline{M}_g$ obtained by attaching a copy of a fixed general pointed elliptic curve at each of the g marked points as in the figure below. We'll loosely refer to the image of u here as the flag locus and to a curve in it as a flag curve. A generalization of this locus plays an important role in the next lecture too.

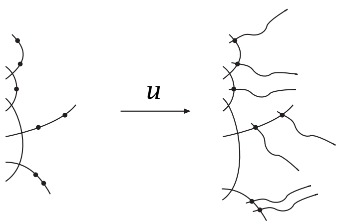


FIGURE 2.8: A typical flag curve

The second space they use is $\overline{M}_{2,1}$, the moduli space of stable one-pointed curves of genus 2 equipped with the map $v : \overline{M}_{2,1} \longrightarrow \overline{M}_g$ obtained by attaching a fixed general smooth one-pointed curve of genus $g - 2$ at the marked point.

It seems to be rather common that loci of special curves in $\overline{M}_g$ meet $v(\overline{M}_{2,1})$ only along the closure $W \subset \overline{M}_{2,1}$ of the locus in which the marked point is a Weierstrass point of the underlying curve. This is the case for both the D^r_s and the Petri divisors mentioned above. Similarly, the curves in $i(\overline{M}_{0,g})$ seem to be rather general. This time D^r_s—but not the more general Petri divisors—misses $u(\overline{M}_{0,g})$ entirely.

We can then find relations on coefficients by applying:

Pullback Lemma 2.9 *Let $D \subset \overline{M}_g$ be an effective divisor, with class*

$$D = a\lambda - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\delta_i \,.$$

1. *If ν^*D is supported on W, then $a = 5b_1 - 2b_2$ and $b_{\mathrm{irr}} = \frac{b_1}{2} - \frac{b_2}{6}$. Further,*
   if we write $\nu^*D = qW$ for some (rational) number q, then $b_2 = 3q$.
2. *If $u^*D = 0$, then $b_i = \frac{i(g-i)}{g-1}b_1$ for $i = 2, \ldots, \lfloor \frac{g}{2} \rfloor$.*

Exercise 2.10: Show that if a divisor D satisfies the relations in both parts of this theorem, then it satisfies the Brill-Noether Ray theorem for some c. *Hint*: Use the second relation to write b_2 in terms of b_1. Then use the first to show that $\dfrac{a}{b_{\mathrm{irr}}} = 6 + \dfrac{12}{g+1}$. Then show that if $a = g + 3$, then $b_{\mathrm{irr}} = \dfrac{g+1}{6}$ and $b_1 = 1$. The remaining coefficients are then immediate from the second set of relations.

Thus, three tasks remain. First, show that the divisors D^r_s meet $\nu(\overline{M}_{2,1})$ only along the closure of the image of W and miss $u(\overline{M}_{0,g})$ entirely. Second, show that the constant of proportionality c in the Brill-Noether Ray theorem is in fact positive. These both follow from the theory of limit linear series: I end this subsection with a few parenthetical comments about what is involved but the reader will have to refer to Chapter 5 of *Moduli of Curves* [32] for all details. Third, we must prove the Pullback Lemma 2.9. This calls on most of the techniques developed up to this point and is sketched in next subsection.

Recall that the *dual graph* of a stable curve has a vertex for each component of the normalization and an edge for each node joining the vertices corresponding to the two pre-images of the node. A node is called interior if the two pre-images lie on the same component. If the graph obtained by removing interior edges from the dual graph is a tree, then the curve is called *tree-like*. If a treelike curve has no interior nodes, the curve is said to be of *compact type* (as its generalized Jacobian is then the product of those of its components): its dual graph is then really a tree.

If the general fiber of a family of curves possesses a linear series with negative ρ the special fiber may or may not carry such a linear series. Conversely a linear series with negative ρ on the special fiber may or may not smooth to the general fiber. The theory of limit linear series uses a study of the ramification of linear series to identify cases in which such specializations and smoothings must exist and in which various related dimensional postulations hold. It then uses these results to show that certain singular stable curves are general in the Brill-Noether-Petri sense, in particular have no limit series with negative ρ.

The first key point is that the treelike curves in D_s^r are limits of smooth curves possessing linear series with negative ρ, so the theory implies that all curves in D_s^r possess "generalized crude limit series" with negative ρ. On the other hand, the theory also shows that no curve in $u(\overline{M}_{0,g})$ possesses a limit series with negative ρ. Hence, no curve in D_s^r can lie in $u(\overline{M}_{0,g})$. For the similar reasons, D_s^r cannot contain any treelike curve in $v(\overline{M}_{2,1} - W)$. But the generic members of the boundary components of $\overline{M}_{2,1}$ are seen in the figure below:

FIGURE 2.11: Generic curves in $\Delta(\overline{M}_{2,1})$

where all components have elliptic normalizations, and these are treelike curves. Thus the locus of non-treelike curves is of codimension > 1 in $v(\overline{M}_{2,1})$, and the intersection of $v(\overline{M}_{2,1} - W)$ with a divisor, were it nonempty, could not consist only of non-treelike curves. This gives the first claim and, in turn, shows that the Brill-Noether Ray theorem must be true for some c using the exercise above.

The second statement shows that the coefficient cb_2 of δ_2 in the Brill-Noether Ray theorem is positive. Since D_s^r is effective, this, in turn, implies that $c > 0$ as required. It is proved by exhibiting, on a general member C of $v(W)$, a limit g_d^r that extends to a codimension 1family of nearby smooth curves thus showing that C lies in D_s^r.

The final statement is proved by expressing the Weierstrass point condition in terms of ramification of the canonical (or a limit canonical) series and then using this characterization to express the locus W in terms of the degeneracy loci of certain maps of associated vector bundles.

Proving the Pullback Lemma

We first apply Lemma 1.26 to the map ν to compute the pullbacks of the basic classes on $\overline{M}_g$. We first find that λ, δ_{irr} and δ_1 pullback to the analogous classes on $\overline{M}_{2,1}$ (the last because on this space $\delta_1 = \delta_{1,\emptyset} + \delta_{1,\{1\}}$). Next $\nu^*(\delta_2) = \delta_{2,\emptyset} + \delta_{0,\{1\}} = 0 - \psi_1 = -\psi$ by our ψ-convention. We could see this directly: since the image of ν lies in δ_2 we need to pull back the normal bundle $\mathcal{O}_{\Delta_2}(\Delta_2)$ which is just the normal bundle to the section Σ_1, or by adjunction $-\psi_1$. The boundary classes δ_i, for $i \geq 3$, pullback to 0 (either by quoting Lemma 1.26 or by directly remarking that $\nu(\overline{M}_{2,1})$ is disjoint from the corresponding loci Δ_i).

With this in hand, let's prove the first part of the Pullback Lemma 2.9. Expressing D^r_s as a linear combination of standard classes as in the Brill-Noether pullback theorem, we see that, in terms of the standard classes on $\overline{M}_{2,1}$ we have $\nu^*(D^r_s) = a\lambda - b_{\text{irr}}\delta_{\text{irr}} - b_1\delta_1 + b_2\psi$. Using the second and third claims, we see that for some q we have

$$a\lambda - b_{\text{irr}}\delta_{\text{irr}} - b_1\delta_1 + b_2\psi = q(3\psi - \lambda - \delta_1)$$

and $b_2 = 3q$ follows immediately by equating ψ coefficients, using the independence of ψ from the other classes.

Since λ, δ_{irr} and δ_1 on $\overline{M}_{2,1}$ are pullbacks of the analogous classes on $\overline{M}_2$, where, by the Genus 2 λ–Formula 1.52

$$\lambda = \frac{1}{10}\delta_{\text{irr}} + \frac{1}{5}\delta_1\,,$$

this relation will continue to hold on $\overline{M}_{2,1}$.

Substituting this for λ and $\frac{b_2}{3}$ for q gives

$$\frac{a}{10} - b_{\text{irr}} = \frac{b_2}{30} \quad \text{and} \quad \frac{a}{5} - b_1 = -\frac{2}{5}b_2$$

from which the relations in the first part of the Pullback Lemma 2.9 follow by solving for a and b_{irr}.

Now we turn to the second part of the Pullback Lemma 2.9. This amounts to finding relations on the classes $u^*\lambda$, $u^*\delta_{\text{irr}}$ and $u^*\delta_i$. The first is easy. On any family $\pi : X \longrightarrow B$ of curves of genus g formed by attaching fixed elliptic tails to curves in $\overline{M}_{0,g}$ at the marked points, the vector bundle $\pi_* \omega_{X/B}$ is trivial. In this case, the Attachment Lemma 1.27 says that it is the direct sum of the bundles $(H^0(E_i, \omega_{E_i}) \otimes \mathcal{O}_B)$. Thus, $u^*\lambda = 0$.

Next, note that by the genus formula any component of a curve in $\overline{M}_{g,0}$ must be smooth and rational, the number of nodes is one less than the number of components, and hence the dual graph is a tree. Thus every node is disconnecting, $u(\overline{M}_{0,g})$ is disjoint from δ_{irr}, and hence $u^*\delta_{\text{irr}} = 0$.

To obtain relations amongst the higher δ_i's we express these classes in terms of the classes ε_i on $\overline{M}_{0,g}$ defined, for $i = 2, \dots, \lfloor g/2 \rfloor$, by

$$\varepsilon_i = \sum_{|P|=i} \delta_{0,P}$$

In other words, a node is an ε_i node if one of the two components of the normalization at the node contains i marked points (and the other $(g-i)$) but *which i points* does not matter and ε_i is the class the closure in $\overline{M}_{0,g}$ of the set of two-component curves illustrated schematically in Figure 2.12.

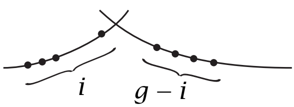


FIGURE 2.12: A General Curve in ε_i

For $i = 2, \dots, \lfloor g/2 \rfloor$, we take ε_i to be the class of the divisor that is the closure in $\overline{M}_{0,g}$ of the set of two-component curves with exactly i of the g marked points on one of the components as illustrated schematically in the figure.

EXERCISE 2.13: The divisors ε_i descend to divisors $\widetilde{\Delta}_i$ on quotient $\overline{M}_{0,\tilde{g}} = \overline{M}_{0,g}/\mathfrak{S}_g$ in which the marked points are unordered. Show that the classes $\widetilde{\Delta}_i$, and hence also the ε_i, are independent.

It's natural to introduce these divisors because, for $i \geq 2$, we have $u^*\delta_i = \varepsilon_i$. This follows either by applying Lemma 1.26 or directly by noting that, in $\overline{M}_g$, the map ν attaches an elliptic tail at each marked point so the images of left and right sides of a curve in ε_i have genera i and $g - i$ respectively.

The final equality to check is that:

$$u^*\delta_1 = -\sum_{i=2}^{\lfloor g/2 \rfloor} \frac{i(g-i)}{(g-1)} \varepsilon_i \,.$$

Admit this for a second. If, then, D is any divisor, given in terms of standard classes as in the Pullback Proposition, it will pull back on $\overline{M}_{0,g}$ to

$$-b_1\left(-\sum_{i=2}^{\lfloor g/2 \rfloor} \frac{i(g-i)}{(g-1)} \varepsilon_i\right) - \sum_{i=2}^{\lfloor g/2 \rfloor} b_i\varepsilon_i \,.$$

If, in addition, D misses $u(\overline{M}_{0,g})$ this pullback must be 0 and equating coefficients immediately gives the claimed relations on the coefficients b_i.

As usual, it suffices to check the claimed relation after restricting to families

$$\pi : C \longrightarrow B, \quad \sigma_1, \ldots, \sigma_g : B \longrightarrow C$$

of stable rational g-pointed curves, where B is a smooth curve missing any inconvenient codimension 2 loci in $\overline{M}_{0,g}$, and transverse to relevant codimension 1 loci in $\overline{M}_{0,g}$. We can thus assume that all reducible fibers of C have exactly two components, the general fiber is a smooth curve, and the total space C is a smooth surface. Fix g pointed elliptic curves (E_k, p_k), and let $C' \longrightarrow B$ be the family obtained by attaching a copy of $B \times E_k$ along σ_k and $B \times p_k$. The family $C' \longrightarrow B$ lies in the g-fold self-intersection of the normal crossing divisor δ_1, and, by our characterization of the normal bundle to the discriminant locus in the versal deformation space, $u^*\delta_1$ is thus the sum of the pullbacks of the normal bundles to the branches.

At the point of Δ_1 corresponding to a fiber C'_b of C', the branch corresponding to the k^{th} node has normal bundle equal to $T_{\sigma_k(b),C_b} \otimes T_{p_k,E}$. Thus it pulls back on B to the normal bundle to the section $\sigma_k(B)$, which we may rewrite as $\pi_*(\sigma_k(B))^2$. Thus

$$u^*\delta_1 = \pi_*\Big(\sum_{k=1}^{g} \sigma_k(B)^2\Big).$$

We may contract the component of each reducible fiber meeting the smaller number of sections (or either component if both components meet $g/2$ sections) to obtain a $\mathbb{P}^1$-bundle $\tilde{\pi} : \tilde{C} \longrightarrow B$ with g sections $\tilde{\sigma}_k : B \longrightarrow \tilde{C}$. These sections meet transversely in groups of i over points of ε_i, and are otherwise disjoint. Thus,

$$\tilde{\pi}_*\Big(\sum_{k=1}^{g} \tilde{\sigma}_k(B)^2\Big) = \pi_*\Big(\sum_{k=1}^{g} \sigma_k(B)^2\Big) + \sum_{i=2}^{\lfloor g/2 \rfloor} i\varepsilon_i\,.$$

On any $\mathbb{P}^1$-bundle the difference of two sections is a linear combination of fibers, and thus has self-intersection 0. Applying this remark to $\tilde{\sigma}_k(B) - \tilde{\sigma}_j(B)$ gives the relation $\tilde{\sigma}_k(B)^2 + \tilde{\sigma}_j(B)^2 = 2\tilde{\sigma}_k(B) \cdot \tilde{\sigma}_j(B)$. Summing over all pairs with $k < j$, we get

$$(g-1)\tilde{\pi}_*\Big(\sum_{k=1}^{g} \tilde{\sigma}_k(B)^2\Big) = 2\tilde{\pi}_*\Big(\sum_{k<j} \big(\tilde{\sigma}_k(B) \cdot \tilde{\sigma}_j(B)\big)\Big) = \sum_{i=2}^{\lfloor g/2 \rfloor} \big(i(i-1)\varepsilon_i\big).$$

The last equality comes from noting that at each point of ε_i there will be $\binom{i}{2}$ pairs of sections meeting. Putting the last three formulas together yields

$$u^*\delta_1 = \sum_{i=2}^{\lfloor g/2 \rfloor} \Big(\frac{i(i-1)}{g-1}\Big)\varepsilon_i - \sum_{i=2}^{\lfloor g/2 \rfloor} (i)\varepsilon_i = -\sum_{i=2}^{\lfloor g/2 \rfloor} \Big(\frac{i(g-i)}{g-1}\Big)\varepsilon_i\,.$$

We have thus verified the final claim and and completed the proof of the Brill-Noether Ray Theorem 2.4.

2.2 Lower bounds: Farkas' results on Koszul divisors

Questions and answers

When $g = 23$, Brill-Noether divisors have slope $\frac{a}{b_{\text{irr}}} = \frac{13}{2}$. We can therefore only conclude that the Kodaira dimension is positive and then only if we find two D^r_s's that have distinct support (which Eisenbud and Harris do but we won't).

For genera $g < 23$, according to the Brill-Noether Ray theorem, $\frac{a}{b_{\text{irr}}} < \frac{13}{2}$ and so the first two parts of the Brill-Noether Ray theorem give no information. For almost 20 years, all known examples suggested that Brill-Noether divisors minimize this ratio amongst all effective ones and led Harris and the author, in [31], to the (incorrect!):

SLOPE CONJECTURE 2.14 *If D is any effective divisor on $\overline{M}_g$, the ratio*

$$\frac{a}{b_{\text{irr}}} \geq 6 + \frac{12}{g+1}.$$

This is known for $g \leq 9$ and for $g = 11$ (see Theorem 2.19)—but recent work of Farkas shows that it is false for all g of the form $2s(s+1)$ with $s \geq 2$. Earlier Farkas and Popa produced the first counterexample for $g = 10$ in [21] and other counterexamples were constructed by Farkas in genera 16 and 22 [17] and by Khosla in genus 21 [45]. Farkas' results suggest that the conjecture fails for all sufficiently large g.

For smaller g, these and related constructions have some other nice consequences. The original genus 10 example in [21] has slope 7 and, by applying a lower bound we'll give in a moment shows that $s_{10} = 7$. More strikingly,

THEOREM 2.15 [19, Theorem 4.1] *There is an effective divisor on $\overline{M}_{22}$ of slope $\frac{17121}{2636} = 6.495\ldots$—hence $\overline{M}_{22}$ is of general type—and there is a virtual divisor $\overline{M}_{23}$ of slope $\frac{470729}{72725} = 6.473\ldots$[1] .*

[1] In the same spirit, [19, Theorem 4.3] lowers the minimal n for which $\overline{M}_{g,n}$ is of general type for many $g \leq 21$.

Definition 2.16: A virtual divisor D is a locus whose expected codimension is 1 but whose actual codimension may be 0.

It's often possible to calculate the coefficients of virtual divisors (defined typically as degeneracy loci of a map between vector bundles) but this yields an unconditional result only if one can show that the locus in question has the expected dimension (i.e. the bundle map is generically non-degenerate). This is Farkas' basic plan. In particular, he proves:

Theorem 2.17 [18, Theorem 1.5] *For any g of the form $s(2s+1)$ with $s \geq 2$, there is an effective divisor on $\overline{M}_g$ of slope strictly less than $6 + \frac{12}{g+1}$.*

Ideas behind the constructions

I want to sketch here the beautiful geometric ideas underlying Farkas' constructions as they subsume essentially all known calculations of effective divisors of small slope. Filling in the details involves a technical tour-de-force well beyond these lectures: [19] gives a precis and [17] and [18] contain the full proofs.

The story starts with the observation in [21] that any divisor of slope *less* than $6+\frac{12}{g+1}$ contains the closure of the locus K_g of genus g curves lying on a K3-surface. This follows from the Effective Dichotomy 2.2 and the exercise below.

Exercise 2.18: Let X be a K3-surface of degree $(2g-2)$ in $\mathbb{P}^g$ that has Picard group of rank 1 and is embedded by a primitive class and let B be a Lefshetz pencil of curves lying on X. Show that $\deg_B(\lambda) = g+1$, $\deg_B(\delta_{\text{irr}}) = 6g+18$ and $\deg_B(\delta_i) = 0$.
Hint: The hypothesis on X makes the last degrees immediate since the pencil can contain no reducible curves and the second follows from the classical Lefshetz pencil formula (cf. [29, p.509]).

Indeed, pencils of this form sweep out a dense locus in the $\mathbb{P}^g$-bundle of curves of genus g over the irreducible 19 dimension moduli space of polarized K3-surfaces of degree $2g-2$ and hence have images in $\overline{M}_g$ dense in K_g.

This immediately allows us to determine s_g for small g as was first noticed by Tan [61]. Indeed, for $3 \le g \le 9$ and $g = 11$, a generic curve of genus g is a hyperplane section of a $K3$ surface X of degree $(2g-2)$ in $\mathbb{P}^g$. The general pencil B of such hyperplane sections has, by Exercise 2.18, slope $6 + \frac{12}{g+1}$ so this is a lower bound for s_g. On the other hand, except in genera 4 and 6, $g+1$ is composite and there exists a Brill-Noether divisor of slope equal to this bound by Brill-Noether Ray Theorem 2.4. Chang and Ran ([6] and [5] had earlier handled these cases and the value of s_{10} follows from the computation of $\overline{K}_{10}$ below. The upshot is:

Theorem 2.19 *For $g \le 11$, $s_g = 6 + \frac{12}{g+1}$ except in the cases where $(g+1)$ is composite when we have $s_4 = \frac{17}{2}$, $s_6 = \frac{47}{6}$ and $s_{10} = 7$.*

This naturally suggests asking when K_g is a divisor and computing its class. The expected dimension of K_g is $19 + g$ so we'd expect it to be a divisor when $g = \frac{23}{2}$ (sic!) but this actually happens when $g = 10$ because, by a theorem of Mukai [51], each curve in K_{10} lies on a 3-dimensional family of K3s. This divisor was first noticed by Cukierman and Ulmer [11] who computed its λ and δ coefficients. Farkas and Popa complete the determination of this divisor as

$$\overline{K}_{10} = 7\lambda - \delta_{\mathrm{irr}} - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5$$

To do this, they successively reinterpret $\overline{K}_{10}$ in several ways. The first description, as the locus of curves C carrying a semi-stable vector bundle E of rank 2 for which $\bigwedge^2(E) = K_C$ and $h^0(C,E) > 7$ has a Brill-Noetherian flavor. The second, as the locus of curves carrying an "exceptional" g^4_{12} for which the multiplication map

$$\mathrm{Sym}^2(H^0(L)) \longrightarrow H^0(L^{\otimes 2})$$

is not a isomorphism is slightly Petrified[2]. Using this description they are able to compute the degrees of $\overline{K}_{10}$ on test curves (similar to those of Example 1.47) obtained by attached a fixed genus $(g-i)$ tail to a moving point on a fixed curve of genus i and find the class above.

Farkas' later work depends on further translations and generalizations of this last interpretation in terms of the Green-Lazarsfeld [28] properties (N_i) of the projective resolution of the ideal of the model of C given by the exceptional g^4_{12}. Let's briefly recall the setup. If I is the ideal of the embedding of C in $\mathbb{P}^r$ by sections of an invertible sheaf L, let $R = \bigoplus_m H^0(C, L^{\otimes m})$ and let

$$0 \longrightarrow E_{r+1} \longrightarrow E_r \longrightarrow \cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R \longrightarrow 0$$

be its minimal resolution by free graded $S = \mathbb{C}[x_0, \ldots, x_r]$-modules, then (C, L) is projectively normal (i.e. all sections of $L^{\otimes n}$ are restrictions of homogeneous polynomials) iff $E_0 = S$—this is condition N_0. If this holds then we can view the E_is for $i > 0$ as giving a free resolution of $I = S/R$. Then property (N_i) is said to hold for $i > 0$ if, for all $j \le i$, E_j is a direct sum of copies of $S(-j-1)$. Thus N_1 is the classical condition that C is cut out by quadrics, N_2 means that all relations amongst these quadrics are generated by linear ones and so on. In terms of the vector bundle M on C defined by

$$0 \longrightarrow M \longrightarrow H^0(C, L) \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0\,.$$

(i.e. M is just the kernel of evaluation of sections of L), one can (cf. [48]) restate (N_i) as the surjectivity, for all $j \ge 1$ of the natural Koszul cohomology map

$$\Big(\bigwedge^{i+1} H^0(L)\Big) \otimes H^0(L^{\otimes j}) \longrightarrow H^0\Big(\big(\bigwedge^{i} M\big) \otimes L^{\otimes(j+1)}\Big)\,.$$

Farkas shows that it suffices to check surjectivity for $j = 1$ above and then globalizes this construction to a map φ of vector bundles

[2] A general curve C of genus 10 carries 42 g^4_{12}s residual to g^1_6s that are pencils of minimal degree on C.

over the compactification $\overline{\mathcal{G}}^r_d$ by limit linear series of the stack $\mathcal{G}^r_d$ of g^r_ds on smooth curves. He then defines $\overline{\mathcal{U}}$ to be the degeneracy locus of this map and $\overline{\mathcal{Z}}$ to be the image of $\overline{\mathcal{U}}$ in $\overline{M}_g$ under the map τ forgetting L. These choices depend on the choice of an $i \geq 0$ as above and a second parameter $s \geq 1$. If, in terms of these, we set $r := 2s + si + i$, $g := (r+1)s$ and $d := r(s+1)$, then the Brill-Noether number $\rho_{g,r,d} = 0$ so we expect the map τ to be finite and the loci $\overline{\mathcal{U}}$ and $\overline{\mathcal{Z}}$ to be (effective) divisors.

Assuming this, he calculates the class of $\overline{\mathcal{Z}}$ and shows that, for $s \geq 2$, it has slope strictly between 6 and $6 + \frac{12}{g+1}$. (For the coefficients, which are quartic polynomials in i, see [19].) Each step in this program is highly non-trivial. Although the coefficients of $\overline{\mathcal{Z}}$ are computed by intersecting with test curves determining the necessary degrees involves computations much more subtle than anything we have done here.

Two cases deserves brief comments. When $s = 1$, we have $g = 2i + 3$, $r = g - 1$ and $d = 2g - 2$ so that $\mathcal{Z}$ is the locus of curves whose canonical bundle fails property (N_i). Applying Voisin's proof of Green's conjecture for generic curves ([64] and [63]), we may identify, at least set-theoretically, $\mathcal{Z}$ with the locus of $(i+2)$-gonal curves which is a Brill-Noether divisor and Farkas' results recover the slope $6 + \frac{12}{g+1}$ computed in the Brill-Noether Ray Theorem 2.4. When $i = 0$, $\mathcal{Z}$ can be described as the virtual class of the locus of curves C carrying a g^r_d for which there is a quadric hypersurface containing the corresponding embedding. Deepak Khosla [45] had earlier computed the class of this $\mathcal{Z}$ by completely different methods and his recent preprint [44] unifies and simplifies many such calculations (e.g. that of $\overline{K}_{10}$) through the introduction of tautological classes on spaces of limit linear series.

And, even after all this work, nothing has been proved unconditionally since no general principle guarantees that φ is not everywhere degenerate and hence that the *virtual* divisor $\overline{\mathcal{Z}}$ is not all of $\overline{M}_g$. Checking this for any pair (s, i) involves producing a suitable pair (C, L) curve for which property (N_i) *does* hold. In the original example of [21] with $(s,i) = (2,0)$ (genus 10) this was done by appealing to earlier work of Mukai [51]. The cases $(2,1)$ and $(2,2)$ (genera 16 and 22) were treated in [17][3]. Khosla [45] handles the case $(3,0)$ where $g = 21$ and [18] extends this to $(s,0)$ for any $s \geq 2$; this gives an infinite set of counterexamples to the Slope Conjecture 2.14 with $g = s(2s+1)$. In [18, Theorem 1.5], pairs (C,L) with the required properties (the pair should be Brill-Noether-Petri general and the model should not lie on a quadric) and are produced by a clever inductive process that I will not go into here except to say that each step involves attaching an elliptic component at several points, obtaining a new curve not lying on a quadric but possibly Petri special, and deforming to a smooth curve to recover the Petri generality. Clearly, there are more beautiful surprises waiting in this area.

2.3 Upper bounds: slopes of effective divisors

The author today has the gnawing sense that in posing the Slope Conjecture 2.14, he was not so much a bit optimistic (naive?) as ass-backwards. The constructions described in the preceding section contradict the conjecture but tend to confirm its philosophy suggesting that the conjecture might "almost" hold. As $g \longrightarrow \infty$, Farkas' divisors have slope $6 + \frac{6}{g}$ + terms of order $g^{(-3/2)}$ and arise as loci of curves carrying a linear series whose free resolution violates property (N_i) (recalled below) for some i, making it tempting to pose the

Question 2.20: Is the slope of every effective divisor on $\overline{M}_g$ at least 6? Are divisors with slopes sufficiently close to 6 loci of curves carrying "exceptional" linear series?

---

[3]The slope computed in genus 22 for $(s,i) = (2,2)$ is $6.5039\ldots$ so this $\overline{Z}$ does *not* show that $\overline{M}_{22}$ is of general type. Instead a different syzygy condition tailored to this genus is used to construct the divisor cited in Theorem 2.15.

In this section, I want to sketch the techniques for finding *lower* bounds for s_g (and hence upper bounds for $\mathrm{NE}^1(\overline{M}_g)$) for *large* values of g in [31]. Here our knowledge is very imperfect and we have to work hard even for that. Heuristic arguments in [31] lead to the estimate $s_g \geq O(\frac{1}{g})$, which is so weak that no one has ever sat down and made the argument rigorous[4]. We remain today in this nearly complete state of ignorance: it is not known if there is a positive constant s such that $s_g \geq s$ for all g. All we can say is that such an s cannot be bigger than 6.

A positive answer would provide a new approach to the Schottky problem since Tai, in work [60] that was the inspiration for the Harris-Mumford Theorem 2.5, proves that there are lots of Seigel modular forms of slope less than any such s. No such form (viewed as defining an effective class on the moduli space on a toroidal partial compactification of the moduli space of abelian varieties) could restrict to an effective class on $\overline{M}_g$ (identified in its turn with the Jacobian locus $\mathcal{J}$) and hence each would have to contain $\mathcal{J}$ and give a "geometric Schottky relation". One can even speculate that these might cut out $\mathcal{J}$. As remarked by Beauville, the classical genus 4 Schottky relation can be interpreted in this way. About the shape of $\mathrm{NE}^1(\overline{M}_g)$ outside the λ-δ-plane essentially nothing is known.

Plan of the construction

The basic idea for producing lower bounds is very simple. Produce effective curves $B \subset \overline{M}_g$ whose deformations sweep out all of $\overline{M}_g$ and compute the degrees of λ and the boundary components on B. By the Effective Dichotomy 2.2, any effective divisor $D = a\lambda - b\delta$ must have $\deg_B(D) = a\deg_B(\lambda) - b\deg_B(\delta) \geq 0$ and hence

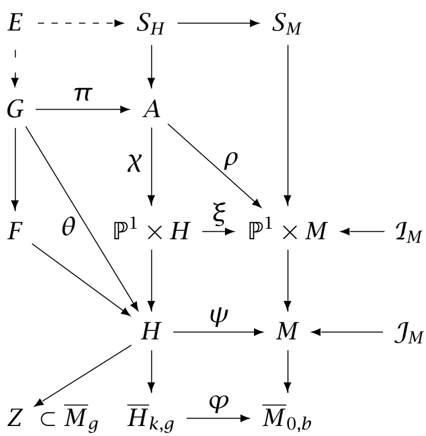
$$s_D := \frac{a}{b} \geq s_B := \frac{\deg_B(\delta)}{\deg_B(\lambda)}$$

[4] Although only because game is not worth the candle.

and since this bound is independent of D it gives a lower bound for s_g as well.

Before sketching the details of how this is worked out, I want to mention one variant of this approach. Instead of looking for a single curve which deforms to cover $\overline{M}_g$, one could look instead for *collections* $\mathcal{B}$ of curves whose union is Zariski dense. Then $\liminf_{B\in\mathcal{B}} s_B$ gives a lower bound for s_g. In his Harvard thesis(cf. [7]), Dawei Chen works out this approach using, as a typical test curve, the family of degree d covers of a fixed elliptic curve B with a single point of ramification of fixed combinatorial type. For suitable choices of the degree d and the ramification type, the corresponding collection $\mathcal{B}$ and there are recursive formulae for s_B. However, the combinatorics of unwinding these formulae is daunting and as I write Chen is able to handle completely only $g = 2$.

(2.21)



The strategy for constructing the necessary curves B is summarized in (2.21) and is, despite appearances, also straightforward[5].

[5]Two caveats are in order. I have tried to keep the notation here consistent with that in earlier sections so it diverges quite a bit from that in [31]; for the better, I hope, as the paper contains several embarassing typos, most glaringly the use of the upper index $\lfloor \frac{g}{2} \rfloor$ where $\lfloor \frac{k}{2} \rfloor$ is wanted in the definitions at the top of page 342.

We begin with any smooth complete curve M, an even number $b = 2a$, and, on $\mathbb{P}^1 \times M$, a b-tuple of sections $\sigma_i \equiv s_M + c_i \cdot f$ with sum $\sigma \equiv b \cdot s_M + c \cdot f$ where f denotes the class of a fiber and s_M the class of a horizontal section. Choosing the sections to be general in the sense that no more than two sections meet at any point, each intersection is transverse and no more than 1 intersection lies in any fiber, the 0-cycle $\mathcal{I}_M$ of intersections will consist of $\mathbf{I}_M = \sum_{i<j}(c_i + c_j) = (b-1)c$ distinct points lying over a reduced cycle $\mathcal{J}_M$ on M[6]. We then blow-up $\mathbb{P}^1 \times M$ at $\mathcal{I}_M$ getting a surface $S_M \longrightarrow M$ which, viewed as a family of stable b-pointed curves of genus 0 with $\mathbf{I}_M$ nodal fibers, gives a map $M \longrightarrow \overline{M}_{0,b}$.

Next fix g, let $k = m - g + 1$ and let $\overline{H}_{k,g}$ be the family of genus g, k-sheeted, simply branched, admissible covers of b-pointed curves of genus 0. We then pull back the tower on the right of (2.21) via the natural covering map (to be recalled in a moment) $\overline{H}_{k,g} \xrightarrow{\varphi} \overline{M}_{0,b}$ to get the tower in the center. In particular, S_H is the blow-up of $\mathbb{P}^1 \times H$ at the set $\mathcal{I}_H := \xi^*(\mathcal{I}_M)$ of singularities of $\tau := \xi^*(\sigma)$ which lies over $\mathcal{J}_H \subset H$.

Over the complement of $\mathcal{J}_H$, the map $S_H \longrightarrow H$ is a nice family of b-sheeted branched covers of $\mathbb{P}^1$ which we would like to complete to a smooth family G of branched covers of a family A of semi-stable b-pointed curves of genus 0 over H. Neither of the obvious extreme approaches to constructing such a family works. The biggest candidate would have as fiber the admissible cover of the b-pointed stable curve corresponding to the image of h in $\overline{H}_{k,g}$ and the smallest (smooth) candidate would have as fiber the stabilization of this cover. The former universal family is shown as E and turns out not to be smooth and to map only rationally to H. The latter would be F and has the defect that it is no longer possible to view its fibers as branched covers of rational curves—i.e. there is no horizontal map to $\mathbb{P}^1 \times H$.

[6]In the rest of this section, we will adopt the convention of using a calligraphic letter $\mathcal{T}$ to denote a combinatorially defined finite *set* and the corresponding bold roman letter $\mathbf{T}$ to denote its *order*.

Both these difficulties can be solved by blowing up the right subset of $\mathcal{J}_H$ to obtain G.

Counting covers

To see what this subset should be, we need to recall the standard description of the data of a simply branched covering. In a fiber $\pi : C \longrightarrow \mathbb{P}^1$ away from $\mathcal{J}_M$, we can do this by choosing a basepoint P not in the set P_i of branchpoints, and a set of pairwise disjoint loops γ_i based at P and having winding numbers δ_{ij} around the points P_j. The hypothesis of simple branching means that we can view the monodromy of π around γ_i as a simple transposition t_i in the symmetric group $\mathfrak{S}_k$ of the fiber of π over P. A collection $\mathbf{t} := (t_1, t_2, \ldots, t_b)$ of transpositions arises in this way from some cover if and only if the relation $\prod_{i=1}^{b} t_i = \mathbf{e}$ holds in $\mathfrak{S}_k$—because $\prod_{i=1}^{b} \gamma_i = \mathbf{e}$ in $\pi_1(\mathbb{P}^1 \setminus \{P_i\})$. This cover is connected if and only if the subgroup of $\mathfrak{S}_k$ generated by the t_i is transitive and two such covers are isomorphic if and only if the corresponding transpositions are simultaneously conjugate in $\mathfrak{S}_k$. Indeed, over $M_{0,b}$, the map φ is an unramified covering whose fiber may be identified with the collection $\mathcal{T}$ of all classes of b-tuples of transpositions up to conjugacy.

The next step is to determine how this covering ramifies over points of $\mathcal{J}_M$ where the base curve lies in Δ_2, the locus of 2-component rational curves with $b - 2$ points on the main component and 2 on the other component[7]. The key to this is to understand what happens to $\mathcal{T}$ as we trace a path Γ_H in H lying over a path Γ_M in M parameterized by $u \in [0, 1]$, that loops once around a point of $\mathcal{J}_M$ where two sections, say σ_1 and σ_2, meet. The diagram on the left below shows what is happening to the points $\sigma_1(u)$ and $\sigma_2(u)$ and that on the right shows what is happening to the based loops γ_1 and γ_2.

[7]This, like most of the geometry in the sequel, is local on M. Variants can be deduced by allowing σ to have less generic singularities (equivalently, by allowing singular covers not mapping to Δ_2), but these seem to give only weaker estimates.

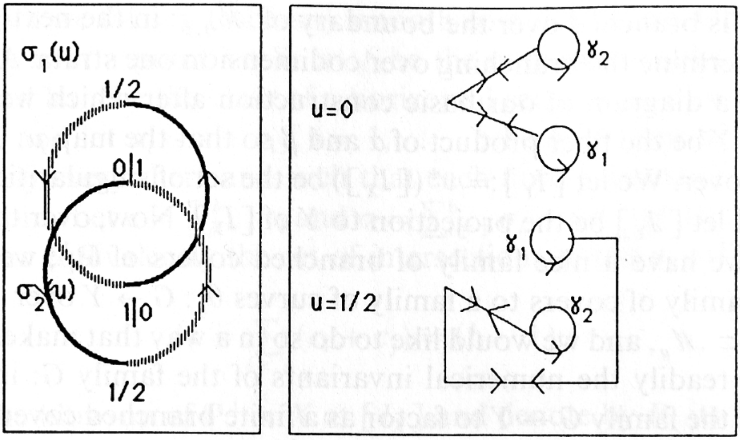


FIGURE 2.22: Branching of $H \xrightarrow{\psi} M$

Homotopically, $\gamma_1(\frac{1}{2}) \sim \gamma_2(0)$ and $\gamma_2(\frac{1}{2}) \sim \gamma_2(0)^{-1} * \gamma_1(0) * \gamma_2(0)$. Hence iterating $\gamma_1(1) \sim \gamma_2(0)^{-1} * \gamma_1(0) * \gamma_2(0)$ and $\gamma_2(1) \sim (\gamma_2^{-1} * \gamma_1 * \gamma_2)^{-1}(0) * \gamma_2(0) * (\gamma_2^{-1} * \gamma_1 * \gamma_2)(0)$. In terms of transpositions, this means that (t_1, t_2) becomes (t_1', t_2') where

$$t_1' = t_2^{-1} t_1 t_2 \text{ and } t_2' = (t_2^{-1} t_1 t_2)^{-1} t_2 \, (t_2^{-1} t_1 t_2)\,.$$

At this point, we need to partition $\mathcal{I}_H$ into subsets $\mathcal{I}_\Pi$ according to the conjugacy class Π of the product $t_1 t_2$ of the pair of transpositions associated to the branch points which meet at each point. We will have Π equal to $(2,2)$, (3) or $\mathbf{e}$, when t_1 and t_2 have, respectively, 0, 1, or 2 letters in common. The preceding computation shows that $H \longrightarrow M$ is unramified over points of type $\mathbf{e}$ and $(2,2)$ and is triply ramified over points of type (3). If we decompose analogously the collection $\mathcal{T}$ of combinatorial data describing points of fibers of $H \xrightarrow{\psi} M$ over points of $M_{0,b}$, we can summarize this discussion in the equation

LEMMA 2.23 $\mathbf{I}_H = \mathbf{T}_{\mathbf{e}} + \mathbf{T}_{(2,2)} + \frac{1}{3}\mathbf{T}_{(3)}$

This decomposition is also the key to picturing the various admissible covers that arise over Δ_2. In these pictures, the 2-pointed component

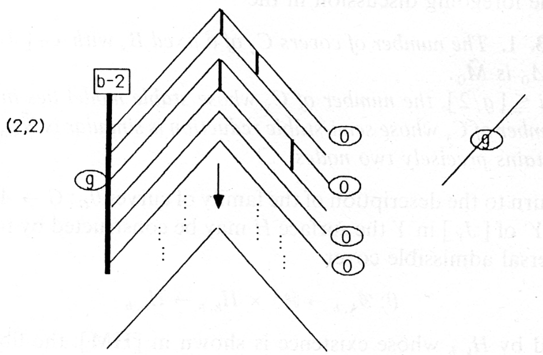


FIGURE 2.24: Cover of type $(2, 2)$

is shown on the *right* (and we'll refer to it as the right component), numbered ovals indicate the genera of components of the normalization, and numbered rectangles count collections of left-side branch points not shown individually. Ramification over the node is determined by the condition that, in an admissible cover of a reducible curve, the product of the permutations on each side must be trivial. On the far right, the *semi*-stable reduction is shown, again with the genus of each component.

For types $(2, 2)$ and (3), a single picture suffices and the semi-stable reduction is always smooth.

Covers of type $\mathbf{e}$ are more interesting because the combinatorics no longer force the left side to be connected. If it is, we get the picture in Figure 2.26. But now the left side may have 2 connected components and the way the genus and the covering degree split between the components may vary as shown in Figure 2.27.

To count these later, we first partition $\mathcal{T}_{\mathbf{e}}$ into subsets $\mathcal{U}_{\text{irr}}$ and $\mathcal{U}_j$ for $2 \leq j \leq \lfloor \frac{k}{2} \rfloor$: $\mathbf{t}$ lies in $\mathcal{U}_{\text{irr}}$ iff the subgroup of $\mathfrak{S}_k$ generated by t_3 to

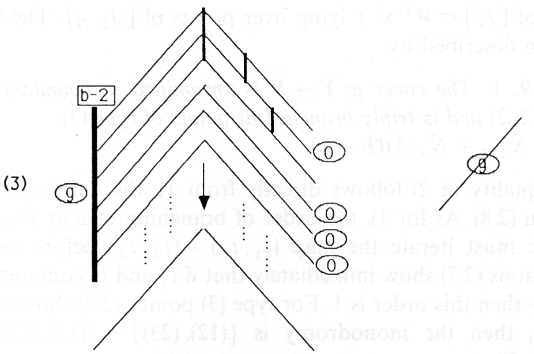


FIGURE 2.25: Cover of type (3)

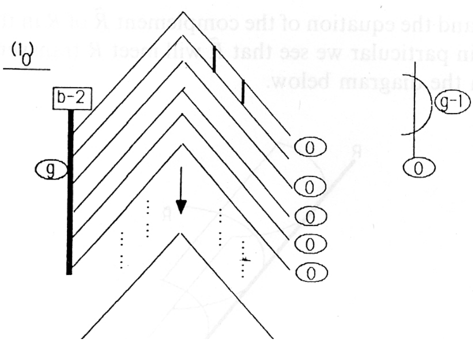


FIGURE 2.26: Irreducible cover of type **e**

t_b acts transitively and in $\mathcal{U}_j$ iff it acts with two orbits of size j and $(k-j)$. We then further partition $\mathcal{U}_j$ into subsets $\mathcal{U}_{j,i}$ for $0 \le i \le g$: **t** lies in $\mathcal{U}_{j,i}$ is the number of t_l with support in the orbit of size j [resp: (k-j)] is $2i + 2j - 2$ [resp: $2(g-i) + 2(k-j) - 2$].

For later use in finding $\deg_Z(\delta)$, we need to reassemble these partitions "by boundary components". For $1 \le i \le \lfloor \frac{g}{2} \rfloor$, let

$$\mathcal{V}_i := \bigcup_{j=2}^{\lfloor \frac{k}{2} \rfloor} (\mathcal{U}_{j,i} \cup \mathcal{U}_{j,g-i}) \ .$$

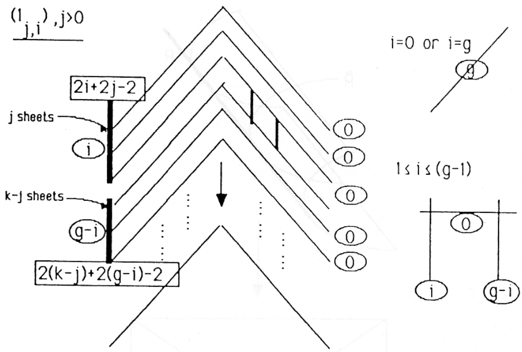


FIGURE 2.27: Reducible cover of type **e**

Finally, to maintain parallelism, let $\mathcal{V}_{\text{irr}} := \mathcal{U}_{\text{irr}}$ and

$$\mathcal{V} := \mathcal{V}_{\text{irr}} \cup \bigcup_{i=1}^{\lfloor \frac{g}{2} \rfloor} \mathcal{V}_i \,.$$

We summarize this analysis in:

LEMMA 2.28 *The number of points in any fiber the map $H \xrightarrow{\psi} M$ lying over a point of $\mathcal{J}_A$ that correspond to admissible covers whose stable model lies in Δ, Δ_{irr} and Δ_i respectively, is $\mathbf{V}$, $\mathbf{V}_{\text{irr}}$, and $\mathbf{V}_i$. The semi-stable reduction of every such cover has exactly 2 nodes.*

A few comments are in order here. The counting functions we have been defining have a venerable history and were known to Hurwitz. They are purely combinatorial in nature and it turns out that recursions for them in terms of values of characters of symmetric groups can be deduced by standard results in character theory. Recursions are needed because what the character theory most naturally counts are the analogues of these functions for covers that are not necessarily connected—that is, with the condition that the transpositions in $\mathbf{t}$ generate $\mathfrak{S}_k$ removed. I'll give no details at all here, simply referring to [31, Section 1].

Finding the right family

Here we want to understand local coordinate pictures of the covers in the previous section well enough to see when we must blow up the corresponding points of $\mathcal{I}_H$ to get the family $G \longrightarrow A$ in (2.21) and when we mustn't. At the same time, we'll gather the information needed to compute $\deg_\lambda(Z)$ in the last subsection.

To prepare for this, suppose that $h \in \mathcal{J}_H$ over which τ_1 and τ_2 meet at a point $(t,h) \in \mathcal{I}_H$ on $\mathbb{P}^1 \times H$. Let U be a neighborhood of h in H and V be product neighborhood of (t,h) over U, both chosen small enough to avoid any other fibers containing singularities of τ. Pick a coordinate u on U and a fiber coordinate v on $\mathbb{P}^1$.

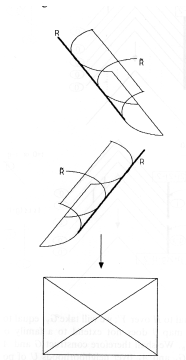


FIGURE 2.29: Local picture at branch point of type $(2,2)$

The $(2,2)$-case is the simplest. Here, although the branch divisor τ has a singularity the ramification divisor R is smooth. It has a component R_i over each τ_i but these lie in two pairs of sheets *disjoint* over V. All that's happening is that two ramification points of the cover $F_y \longrightarrow \mathbb{P}^1$ that are far apart on F happen to line up over $t = 0$. So we can just take $G_U = F_U$ and $A_U = \mathbb{P}^1 \times U$.

If the equation of $\tau_1 + \tau_2$ in V is $u^2 - v^2$, then the equations of the cover G at the points of R lying over (t,y) will be $w^2 = u \pm v$. Thus R

iself will have equation $w = 0$ and its complement R' in the pullback of τ to G will have equation $u \pm w^2 = 0$. As indicated in Figure 2.29, R and R' meet transversally twice over h.

This is also a case in which no extension to a smooth family E of admissible covers can exist because the cover shown in Figure 2.24 has automorphisms given by exchanging sheets of either degree two cover. Alternatively, such a cover would have to have an isolated branch point at the node of the fiber of S_H over y. To get an admissible family would require a base change of order 2 around y to produce τ_1 and τ_2 simply tangent at (t, y), followed by 2 blow-ups to separate them, followed by the blowdown of the first exceptional divisor. But this would leave a surface with an A_1-singularity.

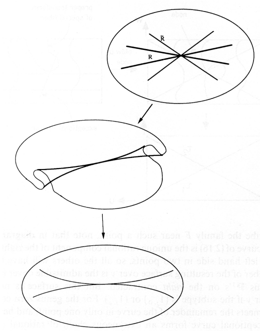


FIGURE 2.30: Local picture at branch point of type (3)

In the type (3) case, where H is triply branched over M, τ_1 and τ_2 no longer meet transversely, but instead have local equations $v = \pm u^3$. However, we again expect from Figure 2.25, that the family F of stable curves should be smooth and expressible as a branched over of $\mathbb{P}^1 \times U$. To confirm this, rescale so that the local equation of $\tau_1 + \tau_2$ in V becomes $27v^2 - 4u^6$ Then, in terms of an additional coordinate w , F has local equation $w^3 - u^2 w + v = 0$ which defines a smooth surface whose fiber over $u = 0$ has the required triple branch point at the origin $v = 0$. Again, we can just take $G_U = F_U$ and $A_U = \mathbb{P}^1 \times U$.

Here, the ramification divisor of $G \longrightarrow A$ has local equation

$$\frac{\partial v}{\partial w} = -3w^2 + u^2 = 0$$

so consists of two smooth arcs meeting transversely and the complement R' of R in the inverse image of the branch divisor τ has equation

$$\frac{27v^2 - 4u^6}{(-3w^2+u^2)^2} = \frac{27(-w^3+u^2w)^2 - 4u^6}{(-3w^2+u^2)^2} = 3w^2 - 4u^2$$

so in neighborhood on F of the point $u = w = 0$ consists of two smooth arcs meeting the 2 arcs of R transversally, as shown in Figure 2.30.

Here again no smooth family E of admissible covers exists. This time no base change is necessary. Were one made, we'd need to blow up $\mathbb{P}^1 \times U$ three times to separate τ_1 and τ_2 and then blow-down the first *two* exceptional divisors which would leave an A_2-singularity.

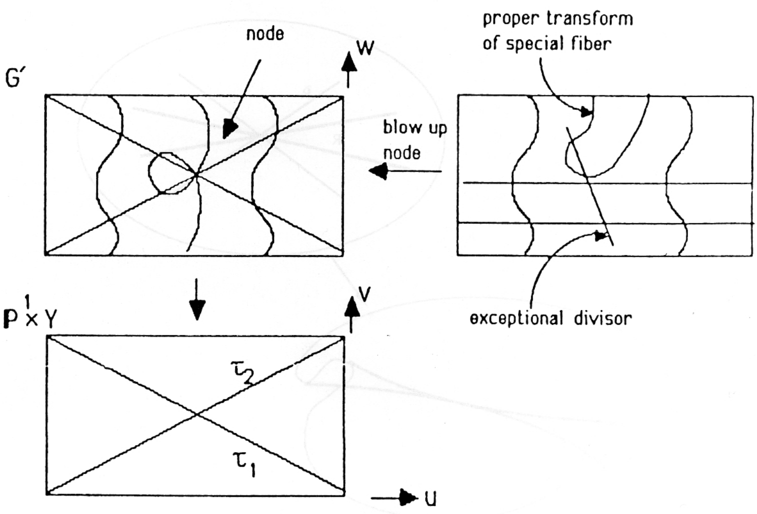


FIGURE 2.31: Local picture at branch point of type **e**

Finally, at points of type **e**, we can complete the family of branched covers of $H \setminus \mathcal{J}_H$ to a family $G' \longrightarrow \mathbb{P}^1 \times H \longrightarrow H$. But then, over $V \subset \mathbb{P}^1 \times U$ we will have $k-2$ smooth sheets and a component with local equation $w^2 = v^2 - u^2$—in other words, the total space G' has an A_1-singularity. However, if we blow up the point $(t, h) \in V$ and its inverse image in G' we arrive a family of admissible covers $E_U \longrightarrow \mathbb{P}^1 \times U \longrightarrow U$, as in Figure 2.31, with fibers as in Figure 2.26 and Figure 2.27 and with *smooth* total space.

Here we take $G_U = E_U$ and A_U to be the blowup of $\mathbb{P}^1 \times Y$ at (t, y). To get the corresponding semi-stable family F, we must blow down all the rational components on the right hand side *except* the exceptional divisor, and, in the reducible cases of type $(j, 0)$ or (j, g), we must also blow down first the genus 0 component of the left hand side (which meets the rest of the cover in only 1 point) and then the exceptional curve.

Degree calculations

We're now ready to get estimates for s_g by computing the degrees of λ and δ on the curve Z of (2.21).

Theorem 2.32 [31, Theorems 3.1 and 3.14] *The degrees of* λ *and* δ *on* Z *are*

$$\deg_Z(\lambda) = \frac{c}{12}\Big((b-1)(3\mathbf{T_e} + \frac{1}{3}\mathbf{T}_{(3)}) - 3\mathbf{T}\Big) \text{ and } \deg_Z(\delta) = 2(b-1)c\mathbf{V}.$$

Hence,

$$s_Z = \frac{72(b-1)\mathbf{V}}{(b-1)(9\mathbf{T_e} + \mathbf{T}_{(3)}) - 9\mathbf{T}}.$$

We'll outline the calculation for λ below. The claim for δ is immediate from Lemma 2.28, Remark 1.43 and the fact that $\mathbf{I} = (b-1)c$ (which also shows that $\deg_Z(\delta_i) = 2(b-1)c\mathbf{V}_i$) and the claim for s_Z then follows from those for λ and δ using Lemma 2.23.

Estimates for s_g follow by taking k bigger than $\lfloor \frac{g+3}{2} \rfloor$. In this range, the Brill-Noether theorem says that deformations of Z are dense in $\overline{M}_g$ allowing us to apply the Effective Dichotomy 2.2. Mild assumptions about the distribution of the set $\mathcal{T}$—for example, that the proportion of $\mathbf{t}$s with any initial pair (t_1, t_2) of transpositions is independent of the choice of the t_is to first order in k—then lead to the estimate $s_g = O(\frac{1}{g})$.

The unconditional results for small g mentioned at the start of this section are obtained by brute force computations of values of the $\mathbf{T}$s and $\mathbf{V}$s use the recursions for these counting functions in [31, Section 1] to which I refer for all details. When $g >> k$, these recursions also give unconditional asymptotic estimates the most striking of which is

Corollary 2.33 *If $g >> k \geq 2$, then any effective divisor on $\overline{M}_g$ of slope less than $\frac{72}{2k+5}$ contains the k-gonal locus.*

For small k, we can relax the assumption that $g >> k$.

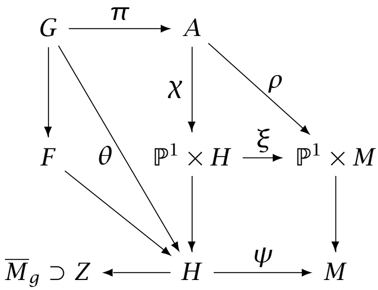
Corollary 2.34 *Let $g \geq 3$ and D be an effective divisor on $\overline{M}_g$.*

1. *If $s_D < 8$, then D contains the hyperelliptic locus.*
2. *If $s_D < 7 + \frac{6}{g}$, then D contains the trigonal locus.*

In particular, the hyperelliptic and trigonal loci are in the base local of all pluricanonical divisors. The bound for the trigonal locus is an improvement due to Tan [61, Theorem 4.1] who uses the classical fact, due to Petri, that trigonal curves C are trisections of rational ruled surfaces $\mathbb{F}_k$ and shows that suitable blowups of general pencils in the linear series $|C|$ give curves filling the trigonal locus and having the indicated slope.

Let's recall our setup in streamlined form.

(2.35)



Since H maps finitely to Z, we can compute $\deg_Z(\lambda)$ on H where we'll use Mumford's Formula 1.44. The last step is to find ω^2 where $\omega := c_1(\omega_{G/H})$ and most of the necessary work has been done in the preceding subsection.

Recall that A is the blowup of $\mathbb{P}^1 \times H$ at the $(b-1)c\mathbf{T_e}$ points of type $\mathbf{e}$. Denote by E the exceptional divisor of this blowup so that

(2.36)
$$E^2 = -(b-1)c\mathbf{T_e}\,.$$

Next let $\hat{\tau}$ be the proper transform of $\chi^*(\tau)$ on A. Note that $\hat{\tau} = \chi^*(\tau) - 2E$ so that

(2.37)
$$\hat{\tau}^2 = (\chi^*(\tau) - 2E)^2 = \rho^*(\sigma)^2 + 4E^2 = 2\mathbf{T}bc + 4E^2$$

and that $\pi_*(R) = \hat{\tau}$ where R is the ramification divisor of the k-sheeted branched cover θ.

To begin with,

$$\omega = \pi^*(\omega_{A/H}) + R = \pi^*(\chi^*(\omega_{\mathbb{P}^1 \times H/H}) + E) + R = \pi^*(-2\chi^*(s_H) + E) + R$$

where s_H is a horizontal section of $\mathbb{P}^1 \times H$. Let's set $L = \pi^*(-2\chi^*(s_H) + E)$ so that we can write ω^2 as $L^2 + 2L \cdot R + R^2$ which we'll compute term-by-term.

First

(2.38)
$$L^2 = \deg(\pi)\left(-2\chi^*(s_H) + E\right)^2 = kE^2$$

since E is orthogonal to the image of χ^*. Next, by push-pull and the fact that τ passes doubly through each point of $\mathcal{T}_{\mathbf{e}}$

(2.39)
$$\begin{aligned} 2L \cdot R &= 2(-2\chi^*(s_H) + E) \cdot \pi_*(R) \\ &= 2(-2\chi^*(s_H) + E) \cdot \hat{\tau} \\ &= 2(-2\chi^*(s_H) + E) \cdot (\chi^*(\tau) - 2E) \\ &= -4(s_H \cdot \tau + E^2) \\ &= -4(\deg(\xi)\, s_M \cdot \sigma + E^2) \\ &= -4(\mathbf{T}c + E^2) \end{aligned}$$

The last term is a bit tricker. To compute it, we use the decomposition of the inverse image of the branch divisor of the covering π as $\pi^*(\hat{\tau}) = 2R + R'$, the fact that $\pi_*(R) = \hat{\tau}$ and the equation

(2.40)
$$R \cdot R' = (b-1)c(2\mathbf{T}_{(2,2)} + \frac{4}{3}\mathbf{T}_{(3)})\,.$$

This last follows from the local descriptions of the preceding section which show that R and R' are disjoint except over points of $\mathcal{I}_H$ of types $(2,2)$ and (3) where they intersect with multiplicities 2 and 4 respectively, and Lemma 2.23 showing that over each of the $b(c-1)$ points of $\mathcal{J}_M$, there lie $\mathbf{T}_{(2,2)}$ points of type $(2,2)$ and $\frac{1}{3}\mathbf{T}_{(3)}$ of type (3). Then,

(2.41)
$$\begin{aligned} R^2 &= R \cdot \frac{1}{2}(\pi^*(\hat{\tau}) - R') \\ &= \frac{1}{2}(\hat{\tau}^2 - R \cdot R') \\ &= \mathbf{T}bc + 2E^2 - (b-1)c(\mathbf{T}_{(2,2)} \end{aligned}$$

by applying (2.37) and (2.40).

Finally, $\deg_A(\delta) = k(b-1)c\mathbf{T}_{\mathbf{e}}$ since the fibers of G over A are smooth except over the points of type $\mathbf{e}$ where each fiber contains k nodes. The claim for $\deg_Z(\lambda)$ now follows by using this, (2.38), (2.39) and (2.41) to compute $\frac{1}{12}(\omega^2_{G/A} + \deg_A(\delta))$ and then simplifying using (2.36) and the tautological relation $\mathbf{T} = \mathbf{T}_{\mathbf{e}} + \mathbf{T}_{(2,2)} + \mathbf{T}_{(3)}$.

2.4 Known effective cones in genus 0

The preceding sections indicate how far we are from understanding even the intersection of the effective cone with the $\lambda - \delta$-plane for $\overline{M}_g$. To my knowledge, there are not even any conjectures about generators for the entire cone $\mathrm{NE}^1(\overline{M}_g)$ or $\mathrm{NE}^1(\overline{M}_{g,n})$ either in terms of geometric loci spanning the extremal rays (assuming these cones are [locally] polyhedral) or in terms of coordinates of these rays in the bases of the Theorem 1.31. However, there are a few cases in genus 0 in which a complete answer is known and in this section I'd like to briefly review these.

The starting point is a paper of Keel and McKernan [43] that must now be the most quoted unpublished work in algebraic geometry and whose ideas inspire many of the results of the next lecture. In it, they consider the space $\overline{M}_{0,\tilde{n}}$ that is as the quotient of $\overline{M}_{0,n}$ by the natural action of $\mathfrak{S}_n$ induced by permuting the marked points. The boundary $\widetilde{\Delta}$ of $\overline{M}_{0,\tilde{n}}$ has components $\widetilde{\Delta}_i$ that are simply the images (cf. Exercise 2.13) of the loci ε_i on $\overline{M}_{0,n}$ of reducible curves with i-points on one side and $(g-i)$ on the other. Now that there is no risk of confusion with classes in $\overline{M}_g$ we will write Δ_i for ε_i and, to simplify notation, adopt the convention in the rest of this section that if explicit indexing is omitted then sums are to be taken over i running from 2 to $\lfloor \frac{g}{2} \rfloor$.

The key result is then the beautiful

Lemma 2.42 *Every $\mathfrak{S}_n$-invariant, effective divisor class D on $\overline{M}_{0,n}$ is an effective sum of the Δ_i*

where by an effective sum of a set of vectors (usually either over a set of effective divisor or over one of effective curve classes), I mean a linear combination of these classes with non-negative coefficients. This immediately implies

Corollary 2.43 *The effective cone* $\mathrm{NE}^1(\overline{M}_{0,\tilde{n}})$ *is simplicial, and is generated by the classes* $\widetilde{\Delta}_i$.

Proof of Lemma 2.42: Any $\mathfrak{S}_n$-invariant D is clearly a linear combination $\sum b_i\Delta_i$ so the point is to show that, if D is effective, then we can take the b_i all non-negative. We do this by induction using test curves. We may assume that D contains no Δ_i since proving the result for the D' that results from subtracting all such contained components will imply the result for D.

As a base for the induction, pick an n-pointed curve $(C, [p_i, \dots, p_n])$ not in the support of D and form a test family with base $B \cong C$ by varying p_n, fixing the other p_i. Since C is not in D, B must meet D non-negatively. On the other hand, $B \cdot \Delta_2 = (n-1)$—there is one intersection each time t_n crosses one of the other t_i—and is disjoint from the other Δ_i. Hence, $b_2 \geq 0$.

Now assume inductively that $b_i \geq 0$. Choose a generic curve

$$C = (C', [p'_1, \dots, p'_i]) \cup C', [p''_1, \dots, p''_{n-i}])$$

in Δ_i in which q' on C' has been glued to q'' on C'' and form the family $B \cong C''$ by keeping q' and the marked points on both sides fixed but varying q'' (Example 1.47 is a model here). As above $B \cdot D \geq 0$, $B \cdot \Delta_j = 0$ unless j is either i or $i+1$. And, as above, $B \cdot \Delta_{i+1} = (n-i)$ (one intersection each time q'' crosses a p''_k), but now B lies *in* Δ_i so to compute $B \cdot \Delta_i$ we use Lemma 1.42. On the left side, the family over B is $C' \times C'$ and the section corresponding to q' has self-intersection 0. On the "right"-side, the family is $C'' \times C'' \cong \mathbb{P}^1 \times \mathbb{P}^1$ blown up at the points where the constant sections corresponding to the p''_k meet the diagonal section corresponding to q'' and hence the proper transform of that section has self-intersection $(2-(n-i))$. The upshot is that $B \cdot D = (n-i)b_{i+1} - (n-i-2)b_i$ completing the induction. □

In fact, this proof shows quite a bit more. It immediately gives the first inequalities in Corollary 2.44 and the others follow by continuing the induction and using the identifications $\Delta_i = \Delta_{n-i}$.

Corollary 2.44 [43, Lemma 4.8] *If $D = \sum b_i \Delta_i$ is an effective class whose support does not contain any Δ_i (or, if D is nef), then $(n - i)b_{i+1} \geq (n-i-2)b_i$ for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ and $ib_{i-1} \geq (i-2)b_i$ for $3 \leq i \leq \lfloor \frac{n}{2} \rfloor$.*

It's natural to hope that we might be able to replace the twiddles in Corollary 2.43 with bars with a bit more work. That we cannot for the first statement, for any $n \geq 6$, is shown by an example of Vermeire [62]. For $n = 6$, work of Rulla [57] and, independently, Hassett and Tschinkel [37] gives the corrected answer. It's necessary to add the components of the loci F_σ of curves fixed by $\sigma \in \mathfrak{S}_n$ where σ runs over all products of three disjoint transpositions[8]. We'll look more closely at the second question for $\overline{M}_{0,n}$ in the next lecture.

I want to close with another Corollary of Lemma 2.42 also due to Keel and kindly communicated to me by Jason Starr. This describes the effective cone of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. It's probably in order to recall, at least telegraphically, a few relevant definitions and facts first. Because it's the only case I'll touch on, I'll stick to maps from curves of genus 0 to $\mathbb{P}^d$. The notes of Fulton and Pandharipande [22] are the best place to start to learn about these now fundamental objects in more generality.

Consider maps $f : (C, [p_1, \ldots, p_n]) \longrightarrow \mathbb{P}^d$ from a connected n-pointed nodal curve of genus g whose image in to $\mathbb{P}^d$ is a curve of degree β and say that a second such map $f' : (C', [p'_1, \ldots, p'_n]) \longrightarrow \mathbb{P}^d$ is isomorphic to f if there is an isomorphism $\tau : C \longrightarrow C'$ of pointed curves such that $f = f' \circ \tau$. Then f itself will have finitely many automorphisms if and only if any component of C *collapsed to a point* by f contains at least 3 "special" points (i.e. marked points or nodes) in which case we say that f is a stable map. So long as they collapse no components, maps can be stable even if their source has *no* marked points. Such

[8]The loci F_σ may also be identified with the pullbacks from $\overline{M}_3$ of the hyperlliptic locus under the map which identifies each pair of transposed points.

maps have a projective Kontsevich moduli stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^d, \beta)$ and a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^d, \beta)$. The simplest example is the space $\overline{M}_{0,0}(\mathbb{P}^d, 1)$ which is nothing other than the Grassmanian of lines in $\mathbb{P}^d$.

The only such space I will consider here is the space $\overline{M}_{0,0}(\mathbb{P}^d, d)$ which is studied further in recent work of Coskun, Harris and Starr ([9] and [10]). A general point of this space has a smooth source curve C with linearly non-degenerate image $f(C) \subset \mathbb{P}^d$ of degree d and hence is nothing more than a rational normal curve. The philosophy is to see $\overline{M}_{0,0}(\mathbb{P}^d, d)$ as the natural compactification of the family of such curves. For example, $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ has an open stratum consisting of plane conics. One boundary divisor arises when the curve C becomes reducible, the map f has degree 1 on each component and image consists of a pair of transverse lines. But there is a second in which the map f degenerates to a double cover of a line in which the image is "virtually marked" with the two branch points. These intersect in a locus of maps from a pair of lines to a single line in which only the image of the point of intersection is "virtually marked". This generalizes: $\mathrm{Pic}(\overline{M}_{0,0}(\mathbb{P}^d, d))$ is generated by effective classes Γ_i, the closure of the locus whose generic map has a domain with two components on which it has degrees i and $d - i$ with $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$, and a class G, the locus where $f(C)$ is degenerate and lies in a proper subspace of $\mathbb{P}^d$.

Lemma 2.45 *A class $D = aG + \sum b_i \Gamma_i$ is effective if and only if $a \geq 0$ and each $b_i \geq 0$.*

All we need to show is that effective classes have positive coefficients. We start with a. Choose a general map $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^d$ for which $g^*(\mathcal{O}(1)) = \mathcal{O}(d)$ (i.e. a generic $d + 1$-dimensional vector space V of degree d polynomials). Then g sends a general pencil B of lines in $\mathbb{P}^2$ to a pencil of rational curves of degree d. The image of a general element of this pencil will be a rational normal curve of degree d, hence non-degenerate, so $g(B) \not\subset G$ and hence $g(B) \cdot G \geq 0$. No element of the pencil will be reducible, hence $g(B) \cdot \Gamma_i = 0$. Since we can make any rational normal curve a member of the pencil by suitably choosing V and B, this family of test curves must meet any effective divisor, in particular, D, non-negatively. So $a \geq 0$.

To handle the b_i, we use a remark of Kapranov [39] that the set K of maps $f \in \overline{M}_{0,0}(\mathbb{P}^d, d)$ whose image contains a fixed set of $d+2$ linearly general points is disjoint from G (by construction) and may be identified with $\overline{M}_{0,d+2}$ (by using the points as the markings) so that points of $\Gamma_i \cap K$ correspond to those of Δ_{i+1}. We can choose K not to lie in D by taking the $(d+2)$-points to lie on a rational normal curve not in D so K must induce an effective class D_K on $\overline{M}_{0,d+2}$. But K does not depend on the ordering of the $d+2$ points so D_K is $\mathfrak{S}_n$-invariant and non-negativity of the b_i follows from Lemma 2.42.

I have only touched on [10]. The reader will find in it a different proof of the result above, sharper in the sense that it produces moving curves dual to the effective classes that are needed in most applications, a stability result for effective cones of $\overline{M}_{0,0}(\mathbb{P}^r, d)$ for $r \geq d$ and intriguing new computations of s_g for $3 \leq g \leq 6$.

Exercise 2.46: Another important effective class on $\overline{M}_{0,0}(\mathbb{P}^d, d)$ is the locus H of maps whose image meets a fixed codimension 2 linear subspace $L \subset \mathbb{P}^d$. In fact, the first determination of the Picard group of $\overline{M}_{0,0}(\mathbb{P}^d, d)$ by Pandharipande [54] showed that it is generated by H and the Γ_I. Show that

$$(d+1)H = 2dG + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} i(d-i)\Gamma_i\,.$$

Chapter 3

Cones of Ample Divisors

This lecture reviews what is known about the ample and nef cones (and, dually, the Mori cone) of the spaces $\overline{M}_{g,n}$. Here, we at least know what we'd like to prove and in many cases we have proved it: the F-Conjecture 3.15 says that all effective curves are combinations of the most obvious ones, the one dimensional strata of the stratification by topological type discussed at the end of Section 1.1 and this is known when $g + n \le 6$ or if $n = 0$ and $g \le 25$. Section 3.2 discusses this material.

These results depend on more classical ampleness results for $\overline{M}_g$ dealing mainly with when combinations of λ and δ are ample. To start, let's recall a few of the earliest results. Historically, the first were due to Arakelov [1] who showed that $\kappa = 12\lambda - \delta$ is ample and that λ is nef. (Exercise 3.10 shows that λ is *not* ample). The GIT construction of $\overline{M}_g$ using Hilbert points of n-canonical models—which works for any $n \ge 5$—realizes it a the Proj of a ring of invariants and hence comes equipped with a natural projective embedding. By tracing through the construction, Mumford [52, Section 5]. showed that this embedding is given by a multiple of $n(12\lambda - \delta) - 4\lambda$. The best result—that $a\lambda - b\delta$ is ample if $a \ge \frac{56}{5}b \ge 0$—is given by taking $n = 5$. On the other hand, $11\lambda - \delta$ has degree 0 on the curves in Example 1.51. The small gap between these results was filled in by Cornalba and Harris who showed that $11\lambda - \delta$ is nef[1]. This result and the basic inequality for the degrees of λ and δ on which it depends will be the main object of Section 3.1 which also briefly reviews some sharpenings by Moriwaki.

3.1 A basic inequality

Suppose we're given a proper flat family $\pi : X \longrightarrow B$ of varieties, and a family of line bundles on the fibers $X_b = \pi^{-1}(b)$ of the family—that is, a line bundle L on X , considered modulo pullbacks of line bundles on B. If L is sufficiently ample, its direct image $\pi_* L$ will be a vector bundle E of some rank $r+1$: we'll assume this from now on. We begin with a theorem that lets us move some of this positivity information down to B. If k is the dimension of the fibers of π, then we set

$$D = (r+1)c_1(L) - \pi^* c_1(E) \subset X \quad \text{and} \qquad F = \pi_*(D^{k+1}) \subset B \ .$$

Theorem 3.1 *Assume that B is one dimensional, and that for a general point $b \in B$ the line bundle $L_b = L|_{X_b}$ is very ample and embeds X_b as a Hilbert stable variety in $\mathbb{P}^r$. Then* $\deg(F) \geq 0$*, i.e.,*

$$(r+1) \cdot \pi_*\left(c_1(L)^{k+1}\right) \geq (k+1) \cdot \pi_*\left(c_1(L)^k\right) \cdot c_1(E).$$

A few words of explanation of the setup here are in order. The class D is chosen to be invariant under tensoring L with the pullback of a line bundle on B, hence to give an invariant of L modulo such pullbacks. Taking the $(k+1)^{\text{st}}$ power of D gives a class of codimension $k+1$ on X so that the push down F is a divisor class on B. Also, since B is one-dimensional, it suffices to exhibit one value of b for which $h^0(X_b, L_B) = r+1$, L_b is very ample and $\varphi_{L_b}(X_b)$ is Hilbert stable. In general, if we assume that these conditions are met for every $b \in B$ (or for all but a finite number), we may deduce that F has nonnegative

[1]Recall that this means that it intersects every effective curve non-negatively.

intersection number with every curve in B and hence that F is nef (lies in the closure of the cone of ample divisors on B).

Cornalba and Harris apply the theorem in the simplest way: given a one-parameter family $\pi : X \longrightarrow B$ of stable curves, with the general fiber X_b smooth and nonhyperelliptic, we take $L = \omega_{X/B}$. Since the degree of L on the fibers of π is $2g-2$ and the degree of L on X is the degree of the line bundle κ on B, we have

$$g\kappa \geq 2(2g-2)\lambda.$$

On the other hand, we know that $\kappa = 12\lambda - \delta$; plugging this in and collecting terms gives:

Corollary 3.2 *If $\pi : X \longrightarrow B$ is any one-parameter family of stable curves, not all hyperelliptic or singular, then the degree of λ and δ on B satisfy*

$$\deg_B(\delta) \leq \left(8 + \frac{4}{g}\right) \cdot \deg_B(\lambda)\,.$$

Because Corollary 3.2 and its consequence Cornalba-Harris Theorem 3.9 are such keys result in this chapter, I am going to reproduce the proofs of them given in Section 6.D of *Moduli of Curves* [32]. We begin with a few reductions. First, for any cover $B' \longrightarrow B$, the divisor class F' associated to the pullback of L to the pullback family $X' = X \times_B B'$ is just the pullback of F to B'. It's thus sufficient to prove the inequality after such a base change; in particular, we may assume, if we like, that the first Chern class $c_1(E)$ is divisible by $r+1$. Next, since the divisor class F was specifically chosen to be invariant under tensoring L with pullbacks of line bundles on B, we may choose a line bundle M on B with first Chern class $c_1(E)/(r+1)$ and replace L by $L \bigotimes \pi^{-1} M^{\vee}$. Thus we may assume that $c_1(E) = 0$ and then what we have to show is that $c_1(L)^{k+1} \geq 0$.

Now consider the natural map

$$\varphi : \operatorname{Sym}^m(E) \longrightarrow \pi_*\left(L^m\right).$$

For sufficiently large values of m, $\mathrm{Sym}^m(E)$ and $\pi_*(L^m)$ will be vector bundles of ranks $O_r(m) = \binom{r+m}{m}$ and $P(m)$ respectively where $P = P_{X_b}$ is the Hilbert polynomial of the fiber X_b of π and the map φ will be generically surjective. We thus have an induced map

$$\psi : W = \Lambda^{P(m)}(\mathrm{Sym}^m(E)) \longrightarrow \Lambda^{P(m)}(\pi_*(L^m))$$

which is likewise generically surjective: since the right-hand side is a line bundle this simply means the map isn't identically zero.

Fix a point $b \in B$ such that on L_b is very ample on X_b and embeds this fiber as a Hilbert stable variety[2] $\overline{X}_b \subset \mathbb{P}^r = \mathbb{P}(E_b^\vee)$, and consider these maps just over that point. The kernel of φ_b is just the m^{th} graded piece of the ideal of $\overline{X}_b$ so the kernel of ψ_b, viewed as a point in the projective space $\mathbb{P}(W_b)$ is just the Hilbert point $[\overline{X}_b]$ of $\overline{X}_b$ in

$$\mathbb{G}(P(m), \mathrm{Sym}^m(E_b)) \subset \mathbb{P}(W_b)\,.$$

Now, by the hypothesis that $\overline{X}_b$ is stable, there exists a homogeneous polynomial f_b of some degree n on the vector space $V := W_b^\vee$, with the properties that

i. f_b is invariant under the action of the group $\mathrm{SL}(E_b)$ on $\mathrm{Sym}^n(V)$;
ii. $f_b([\overline{X}_b]) \neq 0$.

The first of these properties states that: *there is a global holomorphic section f of the bundle $\mathrm{Sym}^n(W)$ whose value at b is f_b*. To see this, observe that, because the vector bundle E has zero first Chern class, we can choose a collection of trivializations $\varphi_\alpha : E_{U_\alpha} \xrightarrow{\cong} \mathcal{O}_{U_\alpha}$ whose transition functions $g_{\alpha\beta}$ take values in $\mathrm{SL}(n, \mathbb{C})$ rather than $\mathrm{GL}(n, \mathbb{C})$. Such trivializations induce trivializations on all the multilinear algebra relatives of E; in particular, we get trivializations $\tilde{\varphi}_\alpha$ of $\mathrm{Sym}^n(W)$ whose transition functions $\tilde{g}_{\alpha\beta}$ preserve f. Thus, if $b \in U_\alpha$ we can simply take f to be given in each coordinate patch by the constant polynomial $f_\alpha = \tilde{\varphi}_\alpha(f_b)$ and the compatibilities $f_\beta = \tilde{g}_{\alpha\beta} f_\alpha$ on the overlaps are automatic.

[2]For an introduction to Hilbert points, see Section1.B of *Moduli of Curves* [32]. The background in G.I.T. and stability needed in what followsis reviewed in Section 4A.

The second property above says simply that *the image of the section f under the map*

$$\mathrm{Sym}^n(\psi) : \mathrm{Sym}^n(W) \longrightarrow \mathrm{Sym}^n\left(\Lambda^{P(m)}(\pi_*(L^m))\right)$$

is nonzero at the point b. In particular, $\mathrm{Sym}^n\left(\Lambda^{P(m)}(\pi_*(L^m))\right)$ has a nonzero global holomorphic section and hence

$$c_1\left(\mathrm{Sym}^n(\Lambda^{P(m)}(\pi_*(L^m)))\right) \geq 0.$$

This is all we really need to know. To start with, this implies that

$$c_1\left(\Lambda^{P(m)}(\pi_*(L^m))\right) = c_1\left(\pi_*(L^m)\right) \geq 0.$$

What is this last quantity? We can try to estimate it by applying the Grothendieck-Riemann-Roch formula to the line bundle L^m on X . Of course, this formula describes, not the Chern class of the direct image, but the alternating sum

$$c_1(\pi_!(L^m)) = \sum_i (-1)^i \cdot c_1(R^i\pi_*(L^m)).$$

In the present circumstances, though, the higher cohomology of L^m vanishes on every fiber of $X \longrightarrow B$, so that the higher direct images of L^m are zero. Grothendieck-Riemann-Roch then tells us that

$$\begin{aligned} c_1(\pi_*L^m) &= [\pi_*(\mathrm{td}(X/B)\cdot \mathrm{ch}(L^m))]_1 \\ &= \pi_*([\mathrm{td}(X/B)\cdot \mathrm{ch}(L^m)]_{k+1}) \\ &= \pi_*\left(\frac{c_1(L^m)^{k+1}}{(k+1)!} + \frac{c_1(L^m)^k}{k!}\cdot \mathrm{td}_1(X/B) + \cdots\right) \\ &= \pi_*\left(m^{k+1}\frac{c_1(L)^{k+1}}{(k+1)!} + m^k\frac{c_1(L^m)^k}{k!}\cdot \mathrm{td}_1(X/B) + \cdots\right). \end{aligned}$$

This last expression is a polynomial in m so, if it's nonnegative for all sufficiently large m, then the leading coefficient must be nonnegative. Thus, as desired, we see that

$$c_1(L)^{k+1} = \deg(f) \geq 0.$$

Exercise 3.3: Show that applying the theorem to a general family $\pi : X \longrightarrow B$ of n-pointed stable curves by applying this theorem to

$\omega_{X/B}(\Sigma)$ where Σ is the usual sum of the n marked sections of π gives the inequality

$$(g + n - 1)\deg_B(\delta - \psi) \le (8g + 10n + 8)\deg_B(\lambda)\,.$$

Why does setting $n = 0$ in this inequality *not* yield Corollary 3.2?

Exercise 3.4: Find the inequalities on the degrees of λ and δ you get for an arbitrary family $X \longrightarrow B$ of stable curves by applying this theorem to *higher* powers of the relative dualizing sheaf and show they are weaker than Corollary 3.2.

Remark 3.5: The argument leading to Corollary 3.2 actually shows a bit more. Suppose b is a point of the base B where the fiber X_b decomposes into two subcurves C and C' meeting at a point p in a node of type δ_i. Then a differential on C not vanishing at p may be viewed as a section of $\omega_{X/B}|_C$ vanishing simply at p. This section then extends locally over B to a section s of $\omega_{X/B}$ vanishing on C'. If X looks locally like $xx' = t^n$ near p with x and x' local equations for C and C', then locally $s = x's'$ with s a section of $\omega_{X/B}$ not vanishing at p. Tracing this refinement through the argument above gives, the sharper inequality

$$\deg_B(\delta_{\text{irr}} + 2\sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \delta_i) \le \left(8 + \frac{4}{g}\right) \cdot \deg_B(\lambda)\,.$$

See [8, (4.4)] for details.

The Cornalba-Harris Theorem

The indirect way the proof of Theorem 3.1 brings in the stability hypothesis on the generic fiber makes it natural to wonder if this hypothesis is really needed. The answer is yes as is shown by an example in [8]. So, it might seem that there's no hope of extending this inequality to families of hyperelliptic curves where the hypothesis fails. By a minor miracle, however, a completely different analysis yields the same inequality for such families. I have omitted this very pretty part of the story and simply refer to Section 6.C of *Moduli of Curves* [32] where it is recounted in detail. In the next subsection, we'll look briefly at a completely different approach to this inequality, due to Moriwaki, which (almost) handles both cases simultaneously. Here, we'll assume this result and see how to use it to get similar (but weaker) inqualities for families with singular general members.

To set this up, let $X \longrightarrow B$ be a family of stable curves whose general fiber has d nodes. By way of terminology, we'll call those nodes of a fiber X_b that are specializations of the nodes on a general fiber the *general nodes* of X_b, and call those nodes of X_b that aren't limits of nodes on nearby fibers the *special nodes* of X_b. Thus, every fiber will have exactly d general nodes and a finite number will have some special nodes as well.

Let $Y \longrightarrow X$ be the normalization of the total space of X: that is, $Y \longrightarrow B$ is the family whose fiber Y_b over any $b \in B$ is the partial normalization of X_b at its general nodes. After making a base change, we can assume that there are $2d$ sections $\sigma_1, \dots, \sigma_{2d} : B \longrightarrow Y$ whose images Γ_l meet a fiber Y_b in the points lying over the general nodes of the corresponding fiber X_b. Note that the general fiber Y_b of $Y \longrightarrow B$ will be reducible if the general fiber of $X \longrightarrow B$ is. If so, then after a further base change we may assume that Y itself is the disjoint union of a collection of families $Y_i \longrightarrow B$ with connected fibers. The exercise below shows that any fiber of one of the Y_i, together with those marked points $\sigma_i(b)$ lying on it, is a *stable* pointed curve. Finally, we replace each Y_i by its minimal desingularization (so that now each fiber of Y_i is a semistable pointed curve).

Exercise 3.6: Let $(C, p_1, \dots, p_n)$ be a stable n-pointed curve. Let $\pi_S : \widetilde{C}_S \longrightarrow C$ be the partial normalization of C at a set S of nodes. Make each connected component D of $\widetilde{C}_S$ into a pointed curve by marking the points on D that map under π to either a marked point of C or a node lying in S. Show that each such component D is then *stable* as a pointed curve .

We're now ready to describe the degrees of the divisor classes λ and δ on B associated to the family $X \longrightarrow B$ in terms of the corresponding classes λ_i and δ_i associated to the families $Y_i \longrightarrow B$ and the self-intersections $(\Gamma_l)^2$ of the images of the sections $\sigma_l : B \longrightarrow Y_i$. We have

$$\deg(\lambda) = \sum_i \deg(\lambda_i) \quad \text{and} \quad \deg(\delta) = \sum_i \deg(\delta_i) + \sum_l (\Gamma_l)^2 .$$

Given this, what is the largest possible ratio of $\deg(\delta)$ to $\deg(\lambda)$? The first thing to notice is that components $Y_i \longrightarrow B$ whose general fiber has large genus g_i do not help maximize this ratio: for such a component we'll have $\deg(\delta_i) \leq (8 + 4/g_i) \cdot \deg(\lambda_i)$, and the sections Γ_l lying on Y_i will have negative self-intersection, bringing the total degree of δ down further. Components Y_i with fibers of genus 1 do better: we have

$$\deg(\delta_i) = 12 \cdot \deg(\lambda_i);$$

and while we do have to have at least one section Γ_α lying on Y_i, its self-intersection will be simply $-\deg(\lambda_i)$. We can thus make up a family of any genus g with

$$\deg(\delta) = 11 \cdot \deg(\lambda) :$$

just take a constant family $C \times B \longrightarrow B$ of smooth curves of genus $g-1$, with constant section $\Gamma = \{p\} \times B$, and attach any family of semistable curves of genus 1 as in Example 1.47.

Can we do better than 11? Clearly, we can do this only by including components Y_i whose general fiber is rational; so we have to digress a moment to investigate the contributions of these.

Lemma 3.7 *If $S \longrightarrow B$ is a family of rational curves with smooth total space and such that each singular fiber contains exactly two components equipped with n pairwise disjoint sections Γ_l. Define d_j to be the* *number of singular fibers with j of the sections passing through one component and $(n-j)$ passing through the other. Then,*

$$(n-1)\cdot\sum_l \Gamma_l^2 = -\sum_{0\le j\le\lfloor n/2\rfloor} j(n-j)\cdot d_j.$$

This argument is a variant on the one used to prove 2. of the Pullback Lemma 2.9. Define a $\mathbb{P}^1$-bundle $T \longrightarrow B$ by blowing down the component of each singular fiber meeting the smaller number of sections—if both meet $\frac{n}{2}$ pick either to blow down—and let G_l be the image of Γ_l on T. Each singular fiber of type d_j in S gives a fiber of T with j sections meeting. Hence,

$$\sum_{l<m} G_l G_m = \sum_{0\le j\le\lfloor n/2\rfloor} \frac{j(j-1)}{2}\cdot d_j\ .$$

But on a $\mathbb{P}^1$-bundle, the difference $G_l - G_m$ of two sections is numerically equivalent to a sum of fibers so has self-intersection 0. Thus, $G_l^2 + G_m^2 = 2G_l\cdot G_m$ and plugging into the preceding inequality gives

$$(n-1)\cdot\sum_l G_l^2 = \sum_{0\le j\le\lfloor n/2\rfloor} j(j-1)\cdot d_j\ .$$

Passing back to S, each blowup in a fiber of type d_j reduces the self-intersection of the j sections which meet at the point by 1. Hence,

$$(n-1)\cdot\sum_l \Gamma_l^2 = (n-1)\cdot\Big(\sum_l G_l^2 - \sum_{0\le j\le\lfloor n/2\rfloor} j\cdot d_j\Big) = -\sum_{0\le j\le\lfloor n/2\rfloor} j(n-j)\cdot d_j\ .$$

EXERCISE 3.8: Show that the relation in Lemma 3.7 holds even if the family $S \longrightarrow B$ has singular points as follows. Suppose that, at a point $p \in S_b$, S has local equation $xy - t^k$. Check that:

1. after blowing down the component of S_b containing $j \le g+1$ of the points $\sigma_l(b)$, the resulting surface will be smooth at the image of p;

2. the corresponding sections Γ_l will meet pairwise with intersection number k, contributing $k\cdot j(j-1)$ to the sum of the intersections $\Gamma_l\cdot\Gamma_m$; and,

3. to recover the original surface S we must first blow up k times to separate the sections Γ_l passing through the image of p, and then blow down the first $(k-1)$ exceptional divisors lowering from the sum of the self-intersections Γ_l^2 by $k \cdot j^2$.

Returning to our family, suppose $Y_i \longrightarrow B$ is a component with rational general fiber and that n_i of the disjoint sections Γ_l lie on Y_i. Lemma 3.7 gives

$$(n_i - 1) \cdot \sum_l \Gamma_l^2 = - \sum_{0 \le j \le \lfloor n/2 \rfloor} j(n_i - j) \cdot d_j$$

where d_j is the number of singular fibers with j of the n_i sections passing through one component and $n_i - j$ passing through the other. By stability, however, each $n_i \ge 3$, so that *the sum of the self-intersections of the sections Γ_l is less than or equal to minus the number of singular fibers*. A component $Y_i \longrightarrow B$ with rational fiber thus contributes nothing to λ and a negative quantity to δ. The upshot is that the ratio of 11 obtained by having elliptic tails is the best we can do. We have thus proved the:

Cornalba-Harris Theorem 3.9 *For any positive integers a and b, the divisor class $a\lambda - b\delta$ is ample on $\overline{M}_g$ if and only if $a > 11b$.*

The following exercise is a warning against the temptation to conclude that λ itself is ample on $\overline{M}_g$: that, in other words, we can let $b = 0$ above.

Exercise 3.10: Let $Y \longrightarrow B$ be the family of stable curves obtained by identifying a fixed point on a fixed smooth curve C_1 of genus g_1 with a variable point on a fixed curve C_2 of genus $g_2 = g - g_1$. Show that $\deg_B(\lambda) = 0$ and hence that the linear system given by any multiple of λ contracts the image of B in $\overline{M}_g$.

Moriwaki's refinements

In this section, I'd like to describe briefly a sharpening of the Theorem 3.1 due to Moriwaki [50] which provides necessary *and sufficient* inequalities for a divisor class D to meet curves in $\overline{M}_g$ *with smooth generic fiber* effectively. I should disclaim, however, that all we'll do here is sketch a toy version of his setup and record a few consequences: you'll have to go to Moriwaki's series of papers leading up to [50] to see the general case and the proofs.

To get a first idea of how Moriwaki's setup works, fix a family of complete curves $\pi : X \longrightarrow B$ over a not necessarily complete base B with smooth general fiber and a vector bundle E of rank r on X such that the restriction of E_b to the general fiber of π is a semi-stable bundle. (Recall that E is called semi-stable if the maximum of the ratio $\frac{\deg(F)}{\mathrm{rank}(F)}$ as F runs over all subbundles of E is achieved by taking $F = E$.) Bogomolov's inequality [4] then asserts that $2rc_2(E) - (r-1)c_1(E)^2$ has non-negative degree on X and hence that $\pi_*(2rc_2(E) - (r-1)c_1(E)^2)$ has non-negative degree on B.

Where can we find a bundle E to apply this inequality to? In general, given a family $\pi : X \longrightarrow B$ and a line bundle L on X such that the rank of $H^0(X_b, L|_{X_b})$ is constant—say equal to h, we get a bundle E of rank $(h-1)$ by taking the kernel of evaluation of sections of L. In other words, E is defined by the exact sequence

$$0 \longrightarrow E \longrightarrow \pi^*\pi_*(L) \longrightarrow L \longrightarrow 0$$

which will restict on fibers to

$$0 \longrightarrow E_b \longrightarrow H^0(X_b, L|_{X_b}) \otimes \mathcal{O} \longrightarrow L_b \longrightarrow 0\ .$$

Moriwaki's idea is to take L to be the relative dualizing sheaf $\omega_{X/B}$ and use the following result of Paranjape and Ramanan [55, Corollary 3.5].

Exercise 3.11: Show that, on a smooth fiber X_b, E_b has rank $(g-1)$ and degree $(2g-1)$. Then use Clifford's theorem to show that any sub-linebundle L has $\frac{\deg(L)}{\operatorname{rank}(F)} \leq 2$ with equality if and only if X_b is hyperelliptic. Conclude that E_b is semi-stable—in fact, E_b is even stable if X_b is not hyperelliptic.

Using the Grothendieck-Riemann-Roch formula, it's then straightforward to unwind the inequality. The rank r of E is $(g-1)$. By definition, $c(\omega_{X/B}) = 1 + \gamma$ and $c\big(\pi_*(\omega_{X/B})\big) = 1 + \lambda$ (there are no higher terms since the base B is one-dimensional). Using multiplicativity,

$$c(E) = (1 + \pi^*(\lambda))(1 + \gamma)^{-1} = (1 + \pi^*(\lambda))(1 - \gamma + \gamma^2 + \dots)$$

from which we find immediately that $c_1(E) = \pi^*(\lambda) - \gamma$, $c_1(E)^2 = \gamma^2 - 2\pi^*(\lambda)\gamma$, and $c_2(E) = \gamma^2 - \pi^*(\lambda)\gamma$. We have $\pi_*(\gamma) = \kappa$ by definition and $\pi_*\big(\pi^*(\lambda)\gamma\big) = (2g-2)\lambda$ because γ has degree $(2g-2)$ on each fiber. Plugging all these evaluations in we find that

$$\begin{aligned} & \pi_*\big(2rc_2(E) - (r-1)c_1(E)^2\big) \\ = \; & 2(g-1)(\kappa - (2g-2)\lambda) - (g-2)(\kappa - 2(2g-2)\lambda) \\ = \; & g\kappa - 2(2g-2)\lambda\,. \end{aligned}$$

This should look familiar: it's the same quantity that lead, by plugging in Mumford's Formula 1.44, to the Cornalba-Harris inequality (cf. Corollary 3.2)

$$\Big(8g + 4\Big) \cdot \deg_B(\lambda) \geq g \deg_B(\delta)\,.$$

Thus Moriwaki's approach can be used to recover the Cornalba-Harris Theorem 3.9 without the special analysis of curves lying in the closure $\overline{H}$ of the hyperelliptic locus. However, the sharper results I am about to quote require essentially all of this extra analysis (in particular, the expression for the restriction of λ to $\overline{H}$ in terms of natural classes on that locus): see the last paragraph of this subsection.

Exercise 3.12: Generalize the calculation above to generically smooth families in $\overline{M}_{g,n}$. More precisely, show that if we take $L = \omega_{X/B}(\Sigma)$ where Σ is as usual the sum of the canonical sections Σ_i giving the n marked points, then the corresponding E again has semistable restiction to smooth fibers, and that Bogomolov's inequality recovers the inequality

$$(g+n-1)\deg_B(\delta-\psi) \le (8g+10n+8)\deg_B(\lambda)\ .$$

of Exercise 3.3.

It's at this point that Moriwaki really begins to work. He modifies the bundle E over the locus of reducible curves with a single node, essentially replacing it by the subsheaf F which maps to 0 in the E's of both components, shows this F is still locally free, and then unwinds Bogomolov's inequality for F to see in [50, Theorem 3.2] that:

Theorem 3.13 *Define the Moriwaki divisor MD on $\overline{M}_g$ by*

$$MD = (8g+4)\lambda - g\delta_{\mathrm{irr}} - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} i(g-i)\delta_i\ .$$

Then MD has non-negative intersection with any curve in $\overline{M}_g$ not lying entirely in the boundary.

This leads to the main result ([50, Corollary 4.3]) that we'll need in the next section:

Theorem 3.14 [Moriwaki's Theorem] *A divisor class*

$$D = a\lambda - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} b_i\delta_i$$

has non-negative intersection with every effective curve in $\overline{M}_g$ not lying in the boundary if and only if:

1. $a \ge 0$.
2. $a \ge \frac{8g+4}{g} b_{\mathrm{irr}}$.
3. $a \ge \frac{2g+1}{i(g-1)} b_i$ *for* $i = 1, \ldots, \lfloor \frac{g}{2} \rfloor$

A divisor D satisfying the inequalities can be decomposed as a positive multiple of the Moriwaki divisor MD plus an effective combination of boundary classes. The MD-term meets curves not lying in the boundary effectively by the previous theorem. The contribution from the boundary terms involves the proper intersection of an effective divisor and an effective curve hence must also be non-negative.

To prove the reverse implication, Moriwaki constructs, for each boundary component, a curve B lying in the closure of the hyperelliptic locus which meets that component effectively and avoids all the others. Moreover, for each such curve, the general fiber of the corresponding family is smooth and the special fibers all have a single node. He then shows that the condition that $D \cdot B \geq 0$ is given by inequalities 2. (for δ_{irr}) and 3. (for the higher δ_i) by applying Cornalba and Harris' analysis of divisors on the hyperelliptic locus[3].

3.2 Mori and nef cones

This section reviews what is known and what is conjectured about the full cone of nef divisors on $\overline{M}_{g,n}$ and its dual, the Mori cone of curves. Tackling this problem may seem like a pretty big stretch given how hard we had to work above just to understand the intersection of the nef cone of $\overline{M}_g$ with the $\lambda - \delta$-plane. But this is one of those cases in which generalizing a problem has the effect of making it easier in many ways. The first key novelty is a conjectural *geometric* description of the extremal rays of the Mori cone, the F-Conjecture 3.15. The second is that the form of this conjecture allows us to make the inductive structure of the set of all these spaces as expressed in the forgetful and glueing maps a powerful tool. Essentially, the general case can be reduced to statements in genus 0; unfortunately, these statements seem to be quite hard.

[3] This is why their analysis is still needed even if Corollary 3.2 can be proved without it.

Almost all the reductions and general statements given here are proved in the joint work of Angela Gibney, Seán Keel and the author [25]. These lead, via a computer assisted attack, to unconditional descriptions of the nef and Mori cones for small g and n that are contained in later work of Gibney (see [24] and the joint paper [20] with Farkas).

Conjectures, reductions, consequences, and open problems

The F-Conjecture 3.15 is motivated by a question originally asked by Fulton only in the case $g = 0$ and whose analogue for $n = 0$ was later considered by Faber [15], whence the name. Recall from Section 1.1, that $\overline{M}_{g,n}$ has a stratification whose strata are indexed by topological type (or equivalently, by the graph isomorphism type of the dual graph)—though only curves in the open cell of each stratum have exactly this type. We will abuse language slightly and use the term curve stratum to refer to the closure of a 1-dimensional stratum. Any nef [ample] divisor class on $\overline{M}_g$ must, by Kleiman's criterion, meet any curves stratum non-negatively [positively]. Greed thus makes it tempting to conjecture:

F-Conjecture 3.15 *Every extremal ray of $\overline{M}_{g,n}$ is spanned by a curve stratum. Equivalently, every effective curve in $\overline{M}_{g,n}$ is numerically equivalent to an effective combination of curve strata; or, a divisor on $\overline{M}_{g,n}$ is nef [ample] iff it has non-negative [positive] intersection with all curve strata.*

With this conjecture in view, it's convenient to make a few working definitions

Definition 3.16: The Faber cone of curves is the subcone of the Mori cones spanned by the curve strata. The Faber cone of divisors is dual to the Faber cone of curves. A Faber curve or Faber divisor is one that lies in the corresponding cone. When the context makes it clear whether curves or divisors are involved, we simply refer to the Faber cone.

The definitions above are deliberately vague about what space is involved so that we can apply them to any space M that has a natural stratification: in the sequel these will be various subloci and quotients of $\overline{M}_{g,n}$. We then say that $F_1(M)$ holds if very extremal ray of M is spanned by a curve stratum and speak of Faber curves and divisors on M.

Before stating what's known, I want to make a series of elementary remarks that turn out to have important consequences in the sequel.

Remark 3.17: 1. The existence of the glueing maps of Definition 1.20 implies that the pullback to $\Delta_{i,I}$ of any line bundle is numerically equivalent to a tensor product of unique line bundles pushed forward from the two factors being glued. The given line bundle is nef on $\Delta_{i,I}$ iff each of the line bundles on the factors is nef.

2. Dually, let B be any curve contained in $\Delta_{i,I}$, and let B' and B'' be its images on the two factors (with multiplicity for the pushforward of cycles) which we also view as curves in $\overline{M}_{g,n}$ by gluing on a fixed curve as usual. Then, B and $B' + B''$ are numerically equivalent.

3. The generic fiber C_b of the family of curves $C \longrightarrow B$ associated to any curve $B \subset \overline{M}_{g,n}$ must consist of components whose modulus varies with b—we call these the moving components of B and their union the moving subcurve—and components which have fixed modulus—we call these the fixed components of B and call their union the fixed subcurve. Arguing inductively using 2, it follows that every B is numerically equivalent to an effective combination of curves whose moving curves are generically irreducible.

4. Finally, if $\theta : \overline{M}_{g',n'} \longrightarrow \overline{M}_{g,n}$ is obtained by by gluing a fixed curve and B' is any curve stratum in $\overline{M}_{g',n'}$, then $B = \theta(B')$ is numerically

equivalent to a curve stratum of $\overline{M}_{g,n}$. Moreover, if D is any divisor on $\overline{M}_g$ and D' is its pullback to $\overline{M}_{g',n'}$, then $D \cdot B = D' \cdot B'$. In particular, if D is a Faber divisor, then so is D'.

The first nice surprise is how easy it is to describe the curve strata of $\overline{M}_{g,n}$—at least up to numerical equivalence—and to compute the degrees of the standard divisor classes on them. By Proposition 1.23, the closure of every stratum is the finite image of a product of spaces M_{g_i,n_i}. But only only $M_{0,3}$ is 0-dimensional and only $M_{0,4}$ and $M_{1,1}$ are 1-dimensional so we must have one factor of the latter type and several of the former. This still leaves an enormous number of *combinatorial* possibilities for the dual graph but we can effectively ignore these by viewing curve strata B as test curves in which a moving component (either $M_{0,4}$ or $M_{1,1}$) is attached to a fixed curve (the various $M_{0,3}$s). Applying the Attachment Lemma 1.27 and Basic Dictionary 1.40, we see that the numerical equivalence class of B is unchanged if we replace the fixed curve by any smoothing of it at the set of nodes *not* on the moving component.

Thus, in the $M_{1,1}$-case where the moving component C has genus 1, we can assume that the fixed component is a smooth curve of genus $g - 1$ and that we are in the situation of Example 1.51 with n additional marked points on the fixed curve. This gives case 1. in Faber Inequalities 3.19 below by applying the degree calculations for that example. We denote the corresponding curve stratum, which sweeps out $\Delta_{1,\emptyset}$, by E.

In the $M_{0,4}$-case, things are a bit more complicated. Now the numerical type of the stratum depends on the genus of and marked points lying on each connected component of the fixed curve and the number of points at which each is attached to the moving component. The connected components of the fixed curve thus determine one of the 5 partitions of the 4 marked points. These are shown schematically on the left of (2) to (6) of Figure 3.18 along with the genera for which each occurs, the possible types of the components (shown in the blue

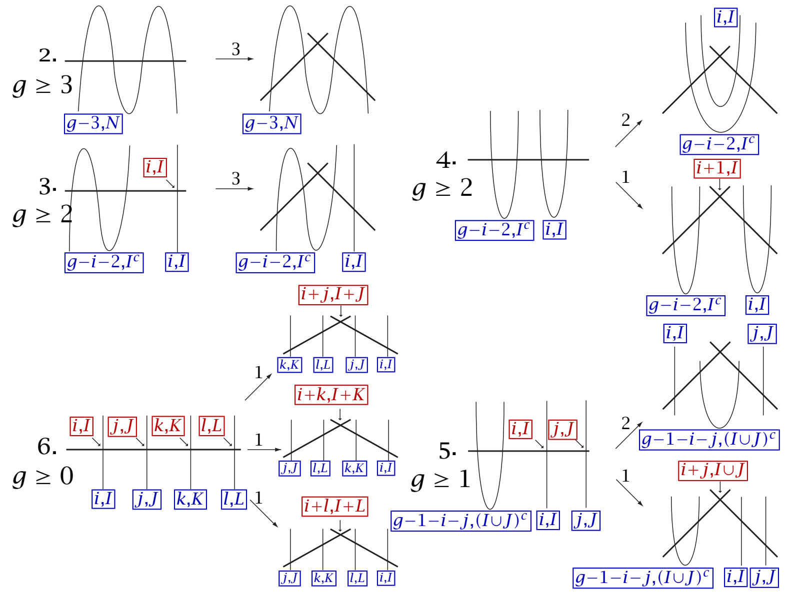


FIGURE 3.18: Curve strata and their degenerations

boxes as a lower case genus and an upper case set of marked points) and the boundary component corresponding to each node (shown in the red boxes except when the node is of irreducible type). The pictures on the right show the 3 special fibers (with multiplicities where 2 or more are numerically equivalent).

The convention, from Chapter 1, of writing $\Delta_{0,\{i\}}$ for $-\psi_i$ to simplify and uniformize statements is used again here and, without comment, in the sequel (although we will also sometimes want to refer to the ψ classes explicitly). In particular, when one of the curves shown is not stable it is to be replaced by its stabilization. Thus, one possibility subsumed in case (3) is that the curve on the left is of type $(g-2, \mathbf{n} \setminus \{i\})$ and the "curve" on the right is collapsed to the i^{th} marked point. The class ψ_i will then have degree -1 on this curve in the 3. of the Faber Inequalities 3.19.

To avoid having to continually "normalize" indices, we define $\mathcal{I}_{g,n}$ to be a set that indexes the boundary components of $\overline{M}_{g,n}$ whose generic element is reducible (plus our pseudoboundaries $\Delta_{0,\{i\}} = -\psi_i$) and allow ourselves to denote elements of $\mathcal{I}_{g,n}$ by pairs (i, I) with the understanding that (i, I) and $(g - i, \mathbf{n} \setminus I)$ denote the same element.

These 5 cases then give the corresponding cases in the Faber Inequalities 3.19 by a straightforward application of Basic Dictionary 1.40 and the generalization to its components of the description of the class δ in Lemma 1.42.

Faber Inequalities 3.19 *Fix a divisor class* $D = a\lambda - b_{\text{irr}}\delta_{\text{irr}} - \sum_{\mathcal{I}_{g,n}} b_{i,I}\delta_{i,I}$ *on* $\overline{M}_{g,n}$ *with the convention that* $\delta_{\text{irr}} = 0$ *if* $g = 0$. *Consider the inequalities*

1. $a - 12b_{\text{irr}} + b_{1,\emptyset} \geq 0$.
2. $b_{\text{irr}} \geq 0$.
3. $b_{i,I} \geq 0$ *for* $0 \leq i \leq g - 2$.
4. $2b_{\text{irr}} \geq b_{i+1,I}$ *for* $0 \leq i \leq g - 2$.
5. $b_{i,I} + b_{j,J} \geq b_{i+j,I\cup J}$
   for $i, j \geq 0, i + j \leq g - 1, I \cap J = \emptyset$.
6. $b_{i,I} + b_{j,J} + b_{k,K} + b_{l,L} \geq b_{i+j,I\cup J} + b_{i+k,I\cup K} + b_{i+l,I\cup L}$
   for $i, j, k, l \geq 0, i + j + k + l = g$, *and* I, J, K, L *a partition of* $\mathbf{n}$.

Then D is a Faber divisor if and only if

- *when* $g \geq 3$, *all of* 1.–6. *hold;*
- *when* $g = 2$, 1. *and* 3.–6. *hold;*
- *when* $g = 1$, 1. *and* 5.–6. *hold;*
- *when* $g = 0$, 6. *holds.*

We now review various partial results in the direction of proving that Faber divisors are nef. All are reductions of an inductive character. We begin with the most important technical result. We say that a divisor meeting all curves B lying in Δ effectively is log-nef.

Faber and log-nef implies nef 3.20 *If $g \geq 2$ or $g = 1$ and $n \geq 2$, a Faber divisor D in $\mathrm{Pic}(\overline{M}_{g,n})$ is nef if and only if its restriction to Δ is nef.*

Using this, the F-conjecture for general g reduces to genus 0 in several ways. The first involves a space which generalizes the space $\overline{M}_{0,\tilde{n}}$ considered in Section 2.4. Define $\overline{M}_{0,n+\tilde{g}}$ to be the quotient of $\overline{M}_{0,n+g}$ by the action of $\mathfrak{S}_g$ on the last g marked points. A point of $\overline{M}_{0,n+\tilde{g}}$ has n ordinary ordered marked points and g *un*ordered (that is, symmetrized) marked points.

We get a map $f_{g,n}$ from $\overline{M}_{0,n+\tilde{g}}$ to a locus $\overline{F}_{g,n}$ in $\overline{M}_{g,n}$ by attaching a fixed pointed curve of genus 1 at each of the g unordered points and the map $f_{g,n}$ is the normalization of image $\overline{F}_{g,n}$. Following the terminology of [12] and [13], we call $\overline{F}_{g,n}$ the flag locus and call curves whose moduli points lie in it flag curves. Figure 2.8 gives a diagram of such a curve when $n = 0$.

Theorem 3.21 *A divisor D on $\overline{M}_{g,n}$ is nef iff D has non-negative intersection with all curve strata and and its restriction to $\overline{F}_{g,n}$ is nef. Conversely, every nef line bundle on $\overline{M}_{0,n+\tilde{g}}$ is the pullback of a nef line bundle on $\overline{M}_{g,n}$. In particular, $F_1(\overline{M}_{0,n+\tilde{g}})$ is equivalent to $F_1(\overline{M}_{g,n})$.*

The proof produces a nef divisor class D on $\overline{M}_g$ which has degree 0 on all curve strata of type 6 (in fact is trivial on $\overline{F}_{g,n}$) and has strictly positive degree on all other curve strata and thus shows that the Mori cone of $\overline{F}_{g,n}$ is a face of the Mori cone of $\overline{M}_{g,n}$.

For $g + n \leq 6$ or $g \leq 11$ and $n = 0$, $F_1(\overline{M}_{0,n+\tilde{g}})$ was known by work of Keel and McKernan [43] so this provided some unconditional evidence for the conjecture in low genus. Recent work of Gibney [24] combines this reduction with numerical techniques for verifying $F_1(\overline{M}_g)$ to extend this range when $n = 0$, in particular, showing that the F-Conjecture 3.15 holds in cases where $\overline{M}_g$ is of general type.

Theorem 3.22 $F_1(\overline{M}_g)$ *holds when* $g \leq 25$.

Curves B whose moving components are not rational generate a subcone N of $\mathrm{Nef}(\overline{M}_g)$. We can sharpen Theorem 3.21 as follows:

Theorem 3.23 *For $g \geq 1$, the subcone N is generated by the curve stratum E of type* 1. *and those curve strata of types* 2.–5. *defined in genus g. Equivalently,* $\mathrm{NE}_1(\overline{M}_{g,n}) = N + (f_{g,n})_*(\mathrm{NE}_1(\overline{M}_{0,n+\tilde{g}}))$.

Next, let us say that, somewhat abusively but with no risk of confusion, that a curve in $\overline{M}_{g,n}$ is rational if all its the components of its normalization are rational. These form a locus $\overline{R}_{g,n} \in \overline{M}_{g,n}$ which, by Lemma 1.8, is the closure of the locus of irreducible g-nodal curves and is the image of the quotient of $\overline{M}_{0,n+2g}$ by the group $\mathfrak{G} \subset \mathfrak{S}_{2g}$ of permutations commuting with the product $(12)(34)\cdots(2g-1\,2g)$ of g transpositions by the map $r_{g,n}$ which identifies the corresponding pairs of marked points (and again normalizes $\overline{R}_{g,n}$). By degenerating all the fixed components, we can find representatives of all the curve strata in 2.–6. of Figure 3.18 lying inside $\overline{R}_{g,n}$. Hence,

Corollary 3.24 *A divisor D on $\overline{M}_{g,n}$ is nef iff its restriction to $\overline{R}_{g,n}$ and to E is nef.*

However, there is no converse here, nor do the curve classes in $\overline{R}_{g,n}$ form a face of $\mathrm{NE}_1(\overline{M}_{g,n})$.

The proofs of these reductions also lead to a few unconditional results of a more ad-hoc nature.

Proposition 3.25 *Let $D = a\lambda - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\delta_i$ be a Faber divisor on M_g such that, for each i either $b_i = 0$ or $b_i \geq b_{\mathrm{irr}}$. Then D is nef.*

In particular, an extremal ray of the Faber cone satisfying these hypotheses must be an extremal ray of the nef cone. Two extremal rays that can be identified in this way are $12\lambda - \delta_{\mathrm{irr}}$ for which the Faber

Inequalities 3.19 of types 1. and 3. are sharp[4] and form a maximal independent set and $10\lambda - 2\delta + \delta_{\mathrm{irr}}$ for which a supporting set of inequalities are given by 1., 3.i for $i > 1$ and 4.i for $i = 1$[5]. Averaging these, we recover the Cornalba-Harris Theorem 3.9[6]—the class $11\lambda - \delta$ is nef—but with the ironic insight that this class does *not* span an extremal ray. All the classes computed by Faber in small genus in [15] satisfy the hypothesis of the Proposition but there are extremal rays of the Faber cone that do not: for example, $30\lambda - 3\delta_{\mathrm{irr}} - 6\delta_1 - 6\delta_2 - 2\delta_3 - 4\delta_4 - 6\delta_5$ on $\overline{M}_{10}$.

The same methods also shed light on morphisms from $\overline{M}_{g,n}$. Recall that a divisor is called semi-ample if some positive multiple is base-point free—such a divisor is necessarily nef. If D is nef, a subvariety Z is exceptional for D if its D-degree is 0, or equivalently if the restriction of D to Z is not big. The exceptional locus $\mathbb{E}(D)$ of D is the union of all D-exceptional subvarieties; if D is semi-ample and big, this is the exceptional locus of the birational map associated to the linear series $|D|$. In her thesis [23] (see also [25]), Gibney showed that

Nef dichotomy 3.26 *If $g \geq 2$ [resp: $g = 1$] and $D \in \mathrm{Nef}(\overline{M}_{g,n})$ then either D is big and $\mathbb{E}(D) \subset \Delta$ or D is the pullback of a nef divisor on $\overline{M}_{g,n-1}$ by the map π forgetting the last marked point [resp: D is the tensor product of pullbacks of nef divisors on $M_{1,P}$ and $M_{1,\mathbf{n}\setminus P}$ by the corresponding forgetful maps $\pi_{P,\mathbf{n}}$ and $\pi_{\mathbf{n}\setminus P,\mathbf{n}}$].*

This has some pretty corollaries:

Corollary 3.27 *If $g \geq 2$, then $\overline{M}_g$ does not fiber over any projective variety and any such fibration of $\overline{M}_{g,n}$ factors through a forgetful map $\pi_{P,\mathbf{n}}$.*

[4] Those of types 5. and 6. are also sharp.

[5] This answers affirmatively the question of p.5 of [15].

[6] Let me emphasize that this does *not* reprove this result since it is used in an essential way in the proof of the Proposition 3.25.

Corollary 3.28 *If $\overline{M}_{g,n} \longrightarrow X$ is any birational morphism to a projective variety X, then the exceptional locus of f lies in Δ. In particular, X is again a compactification of $M_{g,n}$.*

Corollary 3.29 *For $g \geq 5$, the blowdown of the elliptic tails is the only divisorial contraction $f : \overline{M}_{g,n} \longrightarrow X$ of relative Picard number one with X projective.*

We will see in the last lecture that on $\overline{M}_3$ there is a second divisorial contraction, first studied by Bill Rulla [56], associated to the unique extremal ray of type 4. in Figure 3.18.

Next, since $\mathbb{E}(\lambda) = \Delta$, we get the most speaking consequence:

Corollary 3.30 *Any automorphism of $\overline{M}_g$ must preserve Δ.*

The reductions above make it natural to ask how one might attack $F_1(\overline{M}_{0,n})$. A natural question is:

Question 3.31: Is every Faber divisor D on $\overline{M}_{0,n}$ an effective sum of boundary divisors?

This would imply the F-conjecture for $\overline{M}_{0,n}$ and hence for every $\overline{M}_{g,n}$ by an induction which is perhaps the simplest and most telling illustration of the power of the inductive structure of the set of all spaces $\overline{M}_{g,n}$. Indeed, a positive answer would, by the Effective Dichotomy 2.2 reduce showing that a Faber divisor D is nef on $\overline{M}_{0,n}$ to showing that D is nef on every boundary component $\Delta_{0,P}$. By Proposition 1.23, each $\Delta_{0,P}$ is the finite image of a product of $\overline{M}_{0,i}$'s with $i < n$. On the other hand, by 4. of Remark 3.17, D would restrict to a Faber divisor D' on each factor which by induction on n would be nef. Hence D itself would be nef.

Moreover, this question is purely combinatorial and seems, at first, amenable to a computer assisted attack. The combinatorial formulation is

Question 3.32: Let $\mathbf{V}$ be the $\mathbb{Q}$-vector space that is spanned by symbols δ_T for each subset $T \subset \{1, 2, \cdots n\}$ subject to the relations

1. $\delta_T = \delta_{T^c}$ for all T;
2. $\delta_T = 0$ for $|T| \leq 1$; and
3. For each 4 element subset $\{i, j, k, l\} \subset \{1, 2, \cdots n\}$
$$\sum_{i,j \in T,\ k,l \in T^c} \delta_T = \sum_{i,k \in T, j,l \in T^c} \delta_T$$
.

Let $N \subset \mathbf{V}$ be the set of elements $\sum b_T \delta_T$ satisfying

$$b_{I \cup J} + b_{I \cup K} + b_{I \cup L} \geq b_I + b_J + b_K + b_L,$$

for each partition of $\{1, 2, \cdots n\}$ into 4 disjoint subsets I, J, K and L and let $E \subset \mathbf{V}$ be the set of non-negative linear combinations of the δ_T.

Is $N \subset E$?

Two warnings may be offered against too much optimism. First, by [62] not every effective D can be written as a sum of boundaries. Second, the complexity of the combinatorial problem above grows very rapidly with n. The case $n = 6$ can be handled by Porta in a few seconds on a fast PC but attacks on the case $n = 7$ all seem to crash after periods ranging from hours to weeks depending on the machine chosen.

In [24], Gibney formulates[7] the:

G-Conjecture 3.33 *Every Faber class D on $\overline{M}_{0,n}$ is can be written as $D = cK_{\overline{M}_{0,n}} + E$ where $c \geq 0$ and E is an effective sum of boundary divisors.*

Since $K_{\overline{M}_{0,n}} = \sum_{i=2}^{\lfloor n/2 \rfloor} \left(\frac{i(n-1)}{(n-1)} - 2 \right) \Delta_i$, this is a considerably weaker combinatorial assertion than that posed in Question 3.32 but Gibney shows that the G-Conjecture 3.33 for $\overline{M}_{0,n}$ implies $F_1(\overline{M}_{0,n})$ (see the last section of this chapter for details). The numerical results of Theorem 3.22 are attained by verifying this one ray at a time.

[7] Calling it, infelicitously I think, the MF-Conjecture.

Ideas behind the main reductions

We begin with a proof that Faber and log-nef implies nef 3.20 assuming that $g \geq 3$. Fix a divisor class $D = a\lambda - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{\mathcal{I}_{g,n}} b_{i,I}\delta_{i,I}$ on $\overline{M}_{g,n}$. I claim that:

Lemma 3.34 *If $g \geq 1$ and D meets all curve strata of types 1.–5. that are relevant for g effectively, then $D \cdot B \geq 0$ for any curve B not lying in Δ. If $g = 1$, then such a D is linearly equivalent to an effective sum of boundary divisors.*

First, we deal with $g = 1$. In this case, using the relations discussed following Theorem 1.31, we can assume a and all the $b_{0,\{i\}}$ (coefficients of ψ_i) are 0. Then applying the inequalities of type 5. inductively, it follows immediately that $b_{0,I} \geq 0$ for any I and hence that D is equivalent to an effective sum of boundary divisors.

Now assume $g \geq 2$ and define an associated class D' on $\overline{M}_g$ by

$$D' := a\lambda - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \beta_i\delta_i \quad \text{where} \quad \beta_i := \max_{I \subset \mathbf{n}} b_{i,I}$$

Observe that—because we take the β's to be maxima—if the coefficients of D satisfy any one of the *sets* of Faber Inequalities 3.19 of type 1.–5. for $\overline{M}_{g,n}$, then the coefficients of D' satisfy the corresponding inequalities for $\overline{M}_g$. For example in case 5., if I is the subset which computes β_{i+j}, then $\beta_{i+j} = b_{i+j,I} \leq b_{i,\emptyset} + b_{j,I} \leq \beta_i + \beta_j$.

Using the relation $\psi_i = \omega_i + \sum_{i \in I} \delta_{0,I}$ from Exercise 1.30 and writing c_i for $b_{0,\{i\}}$ to simplify notation, we can reexpress D as

$$D = \sum_{i \in \mathbf{n}} c_i\omega_i + \pi^*(D') + E \tag{3.35}$$

where π is the forgetful map to $\overline{M}_g$ and E is the *effective* sum of boundary divisors given by

$$E = \left(\sum_{\substack{I \subset \mathbf{n} \\ \#I \geq 2, \#I^c \geq 2}} \Big(\big(\sum_{i \in I} c_i\big) - b_{0,I} \Big) \delta_{0,I} \right) + \sum_{\substack{(i,I) \in \mathcal{I}_{g,n} \\ i > 0}} (\beta_i - b_{i,I})\delta_{i,I}\,.$$

I claim all three terms in (3.35) meet a curve $C \subset \Delta$ effectively. The classes ω_i are nef by a theorem of Keel [41]. The exercise below handles the middle term.

Exercise 3.36: Show that every Faber class on $\overline{M}_g$ satisfies the inequalities of Theorem 3.14 and hence is log-nef[8].

Finally, the sum defining E is effective. The non-negativity of the coefficient of $\delta_{0,I}$ in E again follows from an induction using the inequalities of type 5. and $c_i = b_{0,\{i\}}$ and that of $\delta_{i,I}$ for positive i from the definition of D'. This proves Lemma 3.34.

From this, Corollary 3.24 follows by an easy induction. We must show that any class D satisfying the inequalities of types 1.–5. must meet effectively any curve B whose moving components are not rational and we proceed by simultaneous induction on g and n. When $g = 0$ or $g = 1$ and $n = 1$ there is nothing to prove so we may suppose $g \geq 2$ or $g = 1$ and $n \geq 2$ and by Lemma 3.34 we may assume that B lies in Δ. First suppose the component containing B is some $\Delta_{i,I}$. As in 4. of Remark 3.17, this induces a decomposition of both B and D and it suffices to show that $D' \cdot B' \geq 0$. But D' is again a Faber divisor and B' again has no rational moving component. If $B \subset \Delta_{\text{irr}}$, we apply the same argument to the normalization at one of the irreducible nodes.

Finally, suppose that D is also nef restricted to $\overline{R}_{g,n}$. In order to see that we can repeat the induction above maintaining this extra hypothesis with respect to the pullback of $\overline{R}_{g,n}$ via the glueing maps involved, we simply have to observe that in 4. of Remark 3.17 we can, by degenerating, choose the fixed curve B'' that we attach to have all components rational without changing its numerical equivalence class.

Next, let's sketch the proof of the reduction to the flag locus, Theorem 3.21. Here the key technical result is

[8] This also follows from the inequality of Remark 3.5 and the analogue for hyperelliptic families. For details, see [25, Lemma 3.5].

Lemma 3.37 *If D is a divisor class satisfying the hypotheses of Lemma 3.34 and let S is a stratum of $\overline{R}_{g,n}$ whose generic member is a curve with no disconnecting nodes, then $D|_S$ is linearly equivalent to an effective sum of boundary divisors and nef divisors.*

Corollary 3.38 *If D is a divisor class on $\overline{M}_{g,n}$ meeting effectively all curve strata of types 1.–5. and B is a curve on $\overline{M}_{g,n}$ whose general member has no moving component that is smooth and rational, then $D \cdot B \geq 0$*

The corollary follows by induction as in the argument proving Corollary 3.24 above. Applying 3. of Remark 3.17, we may assume that the general member of B is irreducible and applying Corollary 3.24 that all components are rational. Then Lemma 3.37 applied to the smallest stratum S whose closure contains B (so that B does not lie in the boundary of S) expresses D as a sum of classes that meet B effectively. Applying Remark 3.17 again, the corollary yields Theorem 3.23.

The proof of Lemma 3.37 proceeds in two steps. As in the proof of Lemma 3.34, we can write

$$D = \sum_{i \in \mathbf{n}} c_i \omega_i + \pi^*(D') + E$$

where π is the forgetful map to $\overline{M}_g$ and E is now an effective sum of boundary divisors of reducible type[9]. Thus, it suffices to prove the lemma replacing D by D' and S by its image under π and we may assume $n = 0$. The image of S is a point if $g = 1$ and is either a point or a curve stratum of type 4. if $g = 2$ (since there are, generically, no disconnecting nodes) so we can assume that $g \geq 3$.

Let C be the stable curve corresponding to a general point of S and let C' be a component of C. By hypothesis, C' comes from a point in the open stratum of some $\overline{M}_{0,k}$ by a glueing map. The k marked points

[9] So E is empty in genus 1.

come equipped with a partition P whose parts are the pairs of points lying over nodes of C' and the subsets lying on the intersection of C' with each connected component of $C \setminus C'$—at least two points lie in each such subset since there are no disconnecting nodes. Observe that each boundary class δ_i on $\overline{M}_g$ will pull back on $\overline{M}_{0,k}$ to an effective sum of classes δ_I where I is a union of parts of P—we'll write $I \succ P$.

The coefficient b_{irr} of δ_{irr} in D must be non-negative by inequality 2. If it is 0, we are done by applying Faber inequalities 3. and 4. If it is positive, we may rescale D so that it equals 1. Then writing Mumford's Formula 1.44 as $-\delta_{\text{irr}} + 12\lambda = \kappa + \sum_{i>0} \delta_i$ and recalling that λ is trivial on $\overline{M}_{0,k}$, we see that D pulls back to

$$\kappa + \sum_{I \succ P} a_I \delta_I$$

with each $a_I \geq -1$. Using the second relation in Exercise 1.35, this can be rewritten as

$$\sum_{I \succ P} \Big(\frac{|I|(k-|I|)}{k-1} - 1 + a_I \Big) \delta_I + \sum_{I \not\succ P} \Big(\frac{|I|(k-|I|)}{k-1} - 1 \Big) \delta_I$$

This expression is almost effective as it stands—$\frac{|I|(k-|I|)}{k-1} \geq 2$ except when $|I| = 2$ and we get $2 - \frac{2}{k-1}$ or when $|I| = 3$ and $k = 6$ when we get $2 - \frac{1}{k-1}$—and the second step in the proof it to use the Four Point Relations 1.34 to massage it into effective form. This is a lengthy but purely combinatorial argument so I will not give it here, referring instead to [25, Lemma 4.4].

To get the converse, consider the class

$$D = a\lambda - b_{\text{irr}}\delta_{\text{irr}} - \sum_{(i,I)} \big(g + n - (i + |I|)\big)\big(i + |I|\big)\delta_{i,I}$$

on $\overline{M}_{g,n}$ in which the ψ classes again appear in the sum (with indexes $(0, \{i\})$). Elementary algebra shows that the intersection of D with any stratum of type 6. is 0. These strata are exactly those inherited from $\overline{M}_{0,n+g}$, and since, by [40], the strata of $\overline{M}_{0,n+g}$ generate its Chow group, this shows that D has trivial pullback to $\overline{M}_{0,n+g}$ and hence to

$\overline{M}_{0,n+\tilde{g}}$. If we now require that $a > 12b_{\text{irr}} - (g+n-1)$ then D meets strata of type 1. *positively*, and if we also require that $b_{\text{irr}} > \binom{n+g}{2}^2$, then it meets strata of types 2.–5. positively. By Theorem 3.21, such a D is nef.

To see that every nef bundle on $\overline{M}_{0,n+\tilde{g}}$ comes from one on $\overline{M}_{g,n}$, first note that, by computations of Faber in [16], the pullback map $f^*_{g,n} : \text{Pic}(\overline{M}_{g,n}) \longrightarrow \text{Pic}(\overline{M}_{0,n+\tilde{g}})$ is surjective and, by construction, D has trivial pullback. Given a class $G \in \text{Pic}(\overline{M}_{0,n+\tilde{g}})$, choose a class E on $\overline{M}_{g,n}$ and such that $G = f^*_{g,n}(E)$. If G is nef then, for large m, $E + mD$ will meet any effective curve *not* pulled back from $\overline{M}_{0,n+\tilde{g}}$ positively (because D does) and hence must itself be nef.

Going a bit further, I claim that $D \cdot B = 0$ if and only if every moving component of the family B is smooth and rational. Since such a B is equivalent to a stratum of the flag locus, this shows that $D^\perp \cap \text{NE}_1(\overline{M}_{g,n}) = \text{NE}_1(\overline{M}_{0,n+\tilde{g}})$. Since we already know that D is trivial on curves coming from the flag locus, the claim follows from

Lemma 3.39 *If $\theta : \overline{M}_{h,p} \longrightarrow \overline{M}_{g,n}$ is a factor of the product decomposition of a stratum of $\overline{M}_{g,n}$ not arising from a smooth rational component of the general member of the stratum, then the exceptional locus of $\theta^*(D)$ lies in the boundary of $\overline{M}_{h,p}$.*

This lemma depends on Corollary 3.27 from Gibney's thesis [23] and leads in turn into the other results on exceptional loci quoted above and I will once again simply reference the proof there or in [25, Lemma 4.8].

Gibney's numerical results

In this subsection, I'll sketch the strategy behind Gibney's numerical attack on the F-Conjecture 3.15 for low values of $g+n$. The key idea goes back to the paper [43] of Keel and McKernan. Consider a sum

$$G = \sum b_I \Delta_I$$

of boundary components of $\overline{M}_{0,n}$ such that $0 \leq b_I \leq 1$ for all I and some $b_I > 0$. In other words, both G and $\Delta - G$ are effective sums of boundaries: I'll say such an G is a *modest* boundary.

F-Ray Theorem 3.40 *If G is as above and R is an extremal ray of $\mathrm{NE}_1(\overline{M}_{0,n})$ for which $(K_{\overline{M}_{0,n}} + G) \cdot R < 0$, then R is spanned by a curve stratum.*

I'll prove this in a moment. First, let's see what consequences it has.

Corollary 3.41 *If the G-Conjecture 3.33 holds on $\overline{M}_{0,k}$ for $k \leq N$ then the F-Conjecture 3.15 holds on $\overline{M}_{0,k}$ for $k \leq N$ and hence, by Theorem 3.21, on $\overline{M}_{g,n}$ whenever $N \geq g + n$.*

Recall that the G-Conjecture 3.33 asserts that every Faber class D on $\overline{M}_{0,k}$ can be written as $D = cK_{\overline{M}_{0,k}} + E$ where $c \geq 0$ and E is an effective sum of boundary divisors. The plan is clear: assume that a Faber divisor D with the shape given in the conjecture meets an extremal ray R negatively and derive a contradiction. We do this by induction on k.

To begin with, choose $0 < d < c$ so that $G = \frac{d}{c}E$ is modest and set $E' = \frac{1-d}{c}E$ which is again an effective sum of boundaries. Then

$$D = (cK_{\overline{M}_{0,k}} + E) = c\big((K_{\overline{M}_{0,k}} + G) + E'\big)\,.$$

If $D \cdot R < 0$, then R is not a Faber ray and we must, by the F-Ray Theorem 3.40, have $(K_{\overline{M}_{0,k}} + G) \cdot R \geq 0$. It follows that $E' \cdot R < 0$.

Now suppose that the ray R is spanned by a curve B. This curve must lie in one of the boundary components in the support of E' which is, via a glueing map, the image of a product of spaces $\overline{M}_{0,j}$ with $j < k$. But the pullback of D to these factors will again be a Faber divisor and hence by induction will be nef. If R is not spanned by a curve, we can choose a curve class B close enough to R that $E' \cdot B < 0$ and then repeat the argument above.

The plan of Gibney's calculations is then straightforward to summarize in principal. First feed the conjectural description of the Faber cone of $\overline{M}_g$ as an intersection of halfspaces implied by Faber Inequalities 3.19 into a package like `lrs` and have it produce the dual description in terms of rays. Then attempt to verify the G-Conjecture 3.33 for each ray: this stage is carried out by a software package called `Nef Wizard` written by Dan Krashen. In practice, to get beyond very low values of g, it's necessary to pullback the calculation to $\overline{M}_{0,\tilde{n}}g$, On this space, heuristics that use previous experience about how earlier nef classes were expressed as sums of boundaries are used to guide the verification for others and certain symmetric averages also streamline the verifications for many D. This work takes up two-thirds of [24] to which I refer to further details.

Now back to the F-Ray Theorem 3.40. The proof here follows the original plan of Keel and McKernan [43, 2.4–2.6] as modified by Farkas and Gibney [20, Theorem 4] and uses some more technical results from minimal model theory that I will just quote. The key step is to show that R lives on some boundary component. Suppose not. Then $R \cdot \Delta_I \geq 0$ for every I and hence $R \cdot K_{\overline{M}_{0,n}} < 0$. Now κ is an ample class with support the full boundary Δ. By an application of the Cone and Contraction theorems [47], the ray R must be spanned by a contractible curve C not lying in Δ, the associated contraction $f : \overline{M}_{0,n} \longrightarrow X$ must be finite on Δ and the relative Picard number of f must be 1. Moreover, by [40] each Δ_I has anti-nef normal bundle—$\Delta_I \cdot B \leq 0$ for any curve $B \subset \Delta_I$.

In this situation, I claim the exceptional locus of f must be the curve C. Given this we reach a contradiction if $n \geq 7$ by applying [46, Theorem 1.14] which estimates the dimension of the space of deformations of C inside $\mathbf{Hilb}(\overline{M}_{0,n})$ as $-K_{\overline{M}_{0,n}} \cdot C + n - 6 \geq n - 6$ and shows that C moves in $\overline{M}_{0,n}$: deformations of C must also lie in the exceptional locus of f. Of the remaining cases, $n = 4$ and $n = 5$ are trivial and $n = 6$ is handled in [20] by a direct verification that the Faber and

Mori cones coincide using the approach of Question 3.32.

To see the claim, assume instead that some irreducible surface S gets mapped by f onto a curve or point. Since Δ has ample support and $f|_\Delta$ is finite, $T := \Delta \cap S$ is non-empty and each $T_I := \Delta_I \cap S$ is an effective $\mathbb{Q}$-Cartier divisor in S which is either empty or a union of components of T. Furthermore, $f|_T$ is finite and f contracts S to an irreducible curve $U \subset f(T)$. Now choose an irreducible component B of T lying in a maximal number of Δ_I. Since the Δ_I have anti-nef normal bundles and Δ has ample support, there must be a Δ_J not containing B and such that $\Delta_J \cdot B > 0$. If B' is a component of $\Delta_J \cap S$, then one of the Δ_I that contains B must, by maximality for B, *not* contain B'. Now Δ_I and Δ_J both meet fibers of f so for a suitable r we must have $\Delta_I - r\Delta_J$ pulled back from X. Now let $V := (\Delta_I - r\Delta_J)|_E$. Our choices mean that $V \cdot B < 0$ and $V \cdot B' \geq 0$. But V is pulled back from U and B and B' are multisections of f so this is a contradiction.

Now that we know that R is pulled back from some glueing map $\theta : \overline{M}_{0,I\cup l} \longrightarrow \overline{M}_{0,n}$, we replace G by the modest divisor $G' = G + (1 - b_I)\Delta_I$. Since Δ_I has anti-nef normal bundle, $(K_{\overline{M}_{0,n}} + G') \cdot R$ is again non-positive. Applying Lemma 1.26, $\theta^*(G') = G'' - \psi_l$ where G'' is again modest and by adjunction $\theta^*(K_{\overline{M}_{0,n}}) = K_{\overline{M}_{0,I\cup l}} + \psi_l$ so $\theta^*(K_{\overline{M}_{0,n}} + G') = K_{\overline{M}_{0,I\cup l}} + G''$. Since $(K_{\overline{M}_{0,I\cup l}} + G'') \cdot R \leq 0$ we can now conclude by induction that R is Faber.

Errata to *Moduli of Curves*

If only Joe and I had devoted to one more reading for content the time we spent just before submission on the copy-editor's concerns over style issues like "red, white, and blue" versus "red, white and blue". Sigh ...

Thanks to those who took the trouble to point out errors. If you find others, a note to me at morrison@fordham.edu would be appreciated.

Page 102, line -10: Replace "g^{th} Fitting ideal" with "$(g-1)^{\text{st}}$ Fitting ideal".

Page 156, line 2: Replace $\frac{\gamma}{2}$ with $\frac{\gamma^2}{12}$.

Page 158, line 13: Replace γ^2 with $\frac{\gamma^2}{2}$.

Page 212, line 5: Replace $(p-1)$ with $(p-i)$..

Page 235, line 2: Replace "open" with "locally closed". Lemma 3.34 shows that the locus U of nodal curves is open in the Hilbert scheme but $\widetilde{\mathcal{K}}$ is closed in U..

Page 235, line 22: Replace $\mathcal{O}_{\mathcal{D}}(1)\big|_{\mathcal{D}_\eta}$ with $\mathcal{O}_{\mathcal{D}}(1)$.

Page 259, line -9: Replace $(d-\sigma)E$ with $(d-\alpha)E$.

Page 267, line -1: Replace "ramification" with "vanishing".

Page 303, line 20: The index in both summations should be α not i.

Page 304, line -11 and -3: We pulled but we forgot to push. Adding the missing π_*'s, these two displays should read

$$(r+1)^{k+1}\pi_*\big(c_1(L)^{k+1}\big) - (k+1)(r+1)^k c_1(E)\pi_*\big(c_1(L)^k\big)$$

and $$(r+1)\cdot\pi_*\big(c_1(L)^{k+1}\big) \geq (k+1)\cdot\pi_*\big(c_1(L)^k\big)\cdot c_1(E).$$

Page 333, line 12: Replace “for such (composite) g” with “when $g+1$ is composite”.

Page 336, line 19: This display should read

$$(V_Y, p) = (rs+r-3, \dots, rs+r-3, rs+r-2, rs+r-1)\,.$$

Page 338, line 12: Replace “Theorem (5.49)” with “Theorem (5.45)”.

Page 341, line 2: Replace “codimension 2” with “codimension 1”.

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